3503 Graph Theory and Combinatorics Notes

Based on the 2013 spring lectures by Dr J Talbot

The Author has made every effort to copy down all the content on the board during lectures. The Author accepts no responsibility what so ever for mistakes on the notes nor changes to the syllabus for the current year. The Author highly recommends that reader attends all lectures, making their own notes and to use this document as a reference only

MATH3503 – Graph Theory and Combinatorics

9 Janu
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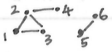
Maths 505.

Course Outline.

Graph theory +

$G=(V,E)$ pair of sets



$V=\{1,2,3,4,5,6\}$
$E=\{\{1,2\},\{1,3\},\{2,3\},\{2,4\},\{5,6\}\}$

- Extremal graph theory, Ramsey theory, colouring

· combinatorics

counting problems ↙ "the number of"

edges in K_5 = # unordered pairs from set $\{1,2,3,4,5\}$ = $\binom{5}{2}=10$

$X=\{1,2,\dots,10\}$. How many cyclic permutations of X are there? 9!

$X=\{1,2,\dots,n\}$. Pick families of subsets of X. Let $\mathcal{A}$ be one such subset, e.g. $\mathcal{A}=\{\{1,2,3\},\{2,4,n\}\}$.
$\mathcal{A}$ is intersecting if $A,B\in\mathcal{A} \Rightarrow A\cap B\neq\emptyset$. No. of subsets of $X=2^n$.
If $3\in\mathcal{A}$, $B^c\notin\mathcal{A}$. so $|\mathcal{A}|\leq 2^{n-1}$, because has at most one of each complementary pairs: $(B, X\setminus B)$
Let $\mathcal{A}=\{A\subseteq \underset{\{1,2,\dots,n\}}{[n]} : 8\in A\}$. Then $|\mathcal{A}|\geq 2^{n-1} \Rightarrow |\mathcal{A}|=2^{n-1}$.

Chapter 1

BASICS.

1.1 Binomial coefficients

$|X|$ denotes the size (or cardinality) of a set X; $k! = k\cdot(k-1)\cdots 2\cdot 1$, $0!=1$ by definition.

$\boxed{\text{Lemma}}$ 1.1 (i) # k-tuples from $X=[n]$ is n^k

(ii) # k-tuples from $X=[n]$ with distinct elements is $n(n-1)\cdots(n-k+1)$.

Proof – (i) n choices for each of k positions // q.e.d.

(ii) n choices for first entry, $n-1$ for second etc... $n-(k-1)=n-k+1$ choices for final k^{th} entry. // q.e.d.

For a given set X, the k-subsets of X are $\binom{X}{k}=\{A\subseteq X: |A|=k\}$. e.g. $\binom{5}{2}=10$ gives the number of ways to pick sets of size 2 from [5]

i.e. $\binom{[5]}{2}=\{12, 13, 14, 15, 23, 24, 25, 34, 35, 45\} \overset{\text{i.e.}}{=} \{\{1,2\},\{1,3\},\dots,\{4,5\}\}$.

$\boxed{\text{Lemma}}$ 1.2 If $|X|=n$, then if $0\leq k\leq n$, $|\binom{X}{k}|=\binom{n}{k}$.

Proof – Each $\underset{\text{(set of size k)}}{\text{k-set}}$ from X corresponds to k! different k-tuples of distinct elements, upon reordering.

Hence, lemma 1.1 $\Rightarrow$ $|\binom{X}{k}| = [n(n-1)\cdots(n-k+1)]\cdot\frac{1}{k!} = \frac{n!}{(n-k)!k!} = \binom{n}{k}$ // q.e.d.

We outline the <u>probabilistic method</u> as a form of proof: idea – we want $_\wedge$ to prove that an example of some mathematical object $_\wedge$ exists. We invent a probabilistic "experiment", where P(the experiment generates a good example) > 0.

Since $0!=1$, $\binom{n}{n}=\binom{n}{0}=1$. We define $\binom{n}{k}=0$ if $k<0$, $k>n$, $k\in\mathbb{Z}$.

$\boxed{\text{Definition}}$ The <u>powerset</u> of a set X, $\mathcal{P}(X)=\{A: A\subseteq X\}$.

$\boxed{\text{Lemma}}$ 1.3 If $|X|=n\geq 0$, $0\leq k\leq n$, then

(i) $|\mathcal{P}(X)|=2^n$, (ii) $\binom{n}{k}=\binom{n}{n-k}$, and (iii) $\binom{n+1}{k}=\binom{n}{k}+\binom{n}{k-1}$

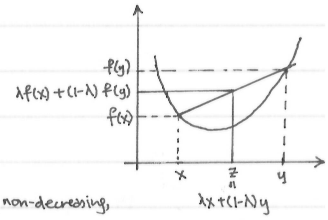
Proof – (i) n elements, in or out $\Rightarrow$ $|\mathcal{P}(X)| = 2\cdots 2 = 2^n$ // q.e.d.

(ii) Algebraically, LHS $=\frac{n!}{k!(n-k)!}=\frac{n!}{(n-k)!k!}=$ RHS // q.e.d. or take $B\mapsto X\setminus B$ as a bijection from $\binom{X}{k}$ to $\binom{X}{n-k}$.

(iii) LHS $=\binom{n+1}{k}=\binom{[n+1]}{k}$ = # k-sets from [n+1] = (# k-sets from [n+1] not containing n+1) + (# k-sets from [n+1] containing n+1)

= (# k-sets from [n]) + (# (k−1)-sets from [n]) $=\binom{n}{k}+\binom{n}{k-1}=$ RHS // q.e.d. (by partitioning).

We want to extend the binomial coefficients from $\mathbb{Z}$ to $\mathbb{R}$: we do this as follows. Let $\overset{x\in\mathbb{R},}{s\in\mathbb{Z}^+}$, $\binom{x}{s}=\begin{cases}\frac{x(x-1)\cdots(x-s+1)}{s!}, & x>s-1\\ 0, & x\leq s-1\end{cases}$



Let a function $f:(a,b)\to\mathbb{R}$ be <u>convex</u>, i.e. $\forall x,y\in(a,b)$, $\lambda\in[0,1]$, then $f(\lambda x+(1-\lambda)y)\leq\lambda f(x)+(1-\lambda)f(y)$.

$\boxed{\text{Lemma}}$ 1.4 If $f:(a,b)\to\mathbb{R}$ is differentiable, $f'(x)$ is non-decreasing on (a,b); then $f(x)$ is convex on (a,b).

Proof – Let $x,y\in(a,b)$, $\lambda\in[0,1]$, $x<y$. If $z=\lambda x+(1-\lambda)y$, apply Mean Value theorem:

$\exists\ \xi_1\in(x,z)$, $\xi_2\in(z,y)$ s.t. $\frac{f(z)-f(x)}{z-x}=f'(\xi_1)$, $\frac{f(y)-f(z)}{y-z}=f'(\xi_2)$. Using the fact that $f'(x)$ is non-decreasing,

$\xi_1<\xi_2 \Rightarrow f'(\xi_1)\leq f'(\xi_2) \Rightarrow \frac{f(z)-f(x)}{z-x}\leq\frac{f(y)-f(z)}{y-z} \Rightarrow f(z)=f(\lambda x+(1-\lambda)y)\leq\lambda f(x)+(1-\lambda)f(y)$ // q.e.d.

$\boxed{\text{Lemma}}$ 1.5 Let $s\geq 1$, $s\in\mathbb{Z}$. Define $\varphi_s:\mathbb{R}\to\mathbb{R}$, $\varphi_s(x)=\binom{x}{s}$, then $\varphi_s(x)$ is convex.

Proof – By induction on s, show $\varphi_s'(x), \varphi_s''(x)\geq 0$ for $x\in(s-1,\infty)$. This is true for $s=1$. We know the fact that $s\varphi_s(x)=(x-s+1)\varphi_{s-1}(x)$.

Differentiate to get $s\varphi_s'(x)=\varphi_{s-1}(x)+(x-s+1)\varphi_{s-1}'(x)\geq 0$ by hypothesis on $s-1$. Similarly, for $\varphi_s''(x)$:

$s\varphi_s''(x)=2\varphi_{s-1}'(x)+(x+s-1)\varphi_{s-1}''(x)\geq 0$ (by induction hypothesis on $s-1$). Hence, $\varphi_s'(x),\varphi_s''(x)\geq 0 \Rightarrow$ by lemma 1.4, $\varphi_s(x)$ is convex // q.e.d.

1.2 Inequalities.

We extend this theory of convex functions to some inequalities.

(Jensen's inequality)

Theorem 1.6 If $\varphi : (a, \infty) \to \mathbb{R}$ is convex, $x_1, \cdots, x_n > a$, $\lambda_1, \cdots, \lambda_n \in [0,1]$, $\sum_{i=1}^{n} \lambda_i = 1$. Then $\varphi(\sum_{i=1}^{n} \lambda_i x_i) \le \sum_{i=1}^{n} \lambda_i \varphi(x_i)$

By induction.

Proof – This is trivially true for $n=1$. It is also true for $n=2$, by definition of a convex function. Now suppose $n \ge 3$. Assume $\lambda_{n-1} + \lambda_n > 0$.

Define $y_i = \begin{cases} x_i, & 1 \le i \le n-2 \\ \frac{\lambda_{n-1}x_{n-1} + \lambda_n x_n}{\lambda_{n+1} + \lambda_n}, & i = n-1. \end{cases}$ $\mu_i = \begin{cases} \lambda_i, & 1 \le i \le n-2 \\ \lambda_{n+1} + \lambda_n, & i = n-1 \end{cases}$

Then $y_1, \cdots, y_{n-1} > a$ and $\mu_1, \cdots, \mu_{n-1} \in [0,1]$, $\sum_{i=1}^{n-1} \mu_i = 1$. Apply inductive hypothesis for $n-1 \Rightarrow \varphi(\sum_{i=1}^{n-1} \mu_i y_i) \le \sum_{i=1}^{n-1} \mu_i \varphi(y_i)$

$\Rightarrow \varphi(\sum_{i=1}^{n} \lambda_i x_i) \le \sum_{i=1}^{n-2} \lambda_i \varphi(x_i) + (\lambda_{n-1} + \lambda_n)\varphi(\frac{\lambda_{n-1}x_{n-1} + \lambda_n x_n}{\lambda_{n+1} + \lambda_n})$. By simple convexity, $\varphi(\sum_{i=1}^{n} \lambda_i x_i) \le \sum_{i=1}^{n} \lambda_i \varphi(x_i)$ // q.e.d.

Corollary 1.7 (Cauchy-Schwarz Inequality)

where $s \ge 1$, $s \in \mathbb{Z}$; $\lambda_1, \cdots, \lambda_n \in [0,1]$ with $\sum_{i=1}^{n} \lambda_i = 1$ and $x_1, \cdots, x_n \ge 0$, $\frac{1}{n}(\sum_{i=1}^{n} x_i)^2 \le \sum_{i=1}^{n} x_i^2$.

Proof – Directly from theorem 1.6, by convexity of $f(x) = x^2$

(Binomial coefficient convexity)

Where $s \ge 1$, $s \in \mathbb{Z}$; $\lambda_1, \cdots, \lambda_n \in [0,1]$ with $\sum_{i=1}^{n} \lambda_i = 1$ and $x_1, \cdots, x_n \ge 0$, $\binom{\frac{1}{n}\sum_{i=1}^{n} x_i}{s} \le \frac{1}{n}\sum_{i=1}^{n}\binom{x_i}{s}$.

Proof – Directly again, by convexity of $f(x) = \binom{x}{s}$ // q.e.d.

Lemma 1.8 If $s \ge 1$ is fixed, then $\frac{(n-s+1)^s}{s!} \le \binom{n}{s} \le \frac{n^s}{s!}$

Proof – $\binom{n}{s} = \frac{n(n-1)\cdots(n-s+1)}{s!}$ Naturally, $n > n-s+1$ etc.

1.3 Graphs

Definition A graph $G = (V, E)$ is a pair of sets, the <u>vertices</u> V and <u>edges</u> E. $E \subseteq \binom{V}{2}$

planar graph.

We denote the vertices and edges of a graph G by $V(G)$ and $E(G)$ respectively.

For examples, refer to handout: Kevin Bacon graph, Erdős graph, internet graph.

Definition The order of a graph is $|V(G)|$, the size of a graph is $|E(G)|$.

The neighbourhood of a vertex $v \in V(G)$ is $\Gamma(v) = \{u \in V(G) : uv \in E(G)\}$.

The degree of vertex $v \in V$, $d(v) = |\Gamma(v)|$.

Note: A vertex is not in its own neighbourhood!

This gives us a lemma concerning the issue of double counting

Lemma 1.9 (Handshake Lemma)

For a graph $G = (V, E)$, $\sum_{v \in V} d(v) = 2|E|$.

Proof – Each edge has 2 endpoints, hence is counted twice in LHS i.e. $\sum_{v \in V} d(v)$. // q.e.d.

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Last time, we established that for a graph $G = (V, E)$, $\sum_{v \in V} d(v) = 2|E|$.

Lemma 1.10 In any graph, the number of vertices of odd degree is even.

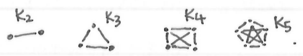
Proof – Let $G = (V, E)$, V be a disjoint union of A and B, $V = A \dot{\cup} B$, $A = \{v : d(v) \text{ odd}\}$, $B = \{v : d(v) \text{ even}\}$.

We know $\sum_{v \in V} d(v) = 2|E|$ is even; and $\sum_{v \in B} d(v)$ is even since it is a sum of even numbers.

Hence, $\sum_{v \in A} d(v) = 2|E| - \sum_{v \in B} d(v)$ is even $\Rightarrow |A|$ is even // q.e.d.

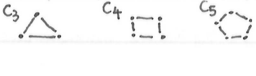
1.4 Special Graphs.

We now define a few special graphs. We have seen earlier that $[n] = \{1, 2, \cdots, n\}$, and we define



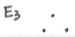
(1) K_n: the complete graph of order $n \ge 2$; with $V = [n]$, $E = \binom{[n]}{2}$.

(2). C_n: the cycle of length $n \ge 3$; with $V = [n]$, $E = \{\{i, i+1\} : i = 1, 2, \cdots, n-1\} \cup \{\{1, n\}\}$.

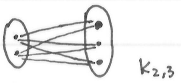


(3) P_n: the path of length n (n edges and $n+1$ vertices); with $V = \{0, 1, \dots, n\}$, $E = \{\{i-1, i\} : i \in [n]\}$.

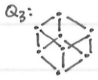
(4) E_n: the empty graph of order n; $V = [n]$, $E = \phi$.



(5) $K_{a,b}$: the complete bipartite graph with classes of size a and b.



(6) Q_n : the (discrete) hypercube of dimension n; $V(Q_n) = \{0,1\}^n$, $E(Q_n) = \{xy \mid x$ and y differ in exactly one coordinate.
$= \{(x_1, \dots, x_n) : x_i \in [0,1]\ \forall i\}$.



Note: $\phi([n]) = \{A : A \subseteq [n]\} \longleftrightarrow \{0,1\}^n$, $A \to \{x_1, \dots, x_n\}$, $x_i = 1$ iff $i \in A$.

1.5 Subgraphs

Let $G = (V,E)$ be a graph, and H be another graph s.t. $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$, then H is a <u>subgraph</u> of G.

We say that H is an <u>induced subgraph</u> of G if $V(H) \subseteq V(G)$ and $E(H) = E(G) \cap \binom{V(H)}{2}$.

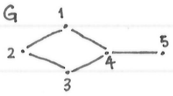
If $G = (V,E)$ is a graph and $A \subseteq V$, then $G[A]$ is the subgraph induced by A:

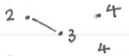
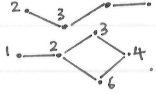
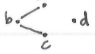
its vertex set is $V(G[A]) = A$ and its edge set is $E(G[A]) = \binom{A}{2} \cap E(G)$.

Graphs G and H are <u>isomorphic</u> $\Leftrightarrow$ $\exists$ bijection $f : V(G) \to V(H)$ s.t. $vw \in E(G) \Leftrightarrow f(v) f(w) \in E(H)$.

We say G contains a <u>copy</u> of H if G has a subgraph isomorphic to H.

e.g. Let the graph G be as depicted on the right. Then, the following cases are



- H_1:  is a subgraph of G, not induced
- H_2:  is an induced subgraph.
- H_3: . H_3 and G are isomorphic
- G contains a copy of H: 

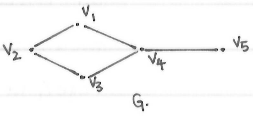
1.6 Components and connectedness.

A <u>path</u> in a graph G is a subgraph isomorphic to P_t for some $t \geq 0$. An <u>x-y path</u> is a path that starts at x and ends at y.

A <u>walk</u> in G is a sequence of vertices (not necessarily distinct) $v_0 v_1 \cdots v_t$ s.t. $v_{i-1} v_i \in E$ for all $i \in [t]$. The walk is <u>closed</u> if $v_0 = v_t$.

A walk in which no edge is used more than once (but vertices may be revisited) is called a <u>tour</u>.

e.g. Consider the graph G on the right:



- $v_1 v_4 v_5$ is a path in G, it is a v_1-v_5 path
- $v_1 v_4 v_5 v_4 v_3$ is a walk in G.
- $v_1 v_4 v_5 v_4 v_3 v_2 v_1$ is a closed walk in G.
- $v_1 v_2 v_3 v_4$ is a tour in G.

$\boxed{\text{Lemma}}$ 1.11 There is an x-y path in G $\Leftrightarrow$ there is a walk from x to y in G.

Proof – ($\Rightarrow$) A path is a walk.

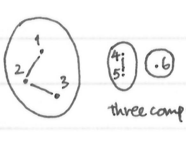
($\Leftarrow$) Take a shortest walk from x to y. If any vertex is revisited we could shorten this walk. Hence, it is a path// q.e.d.

$\boxed{\text{Lemma}}$ 1.12 Define a relation $\sim$ on $V(G)$ by $v \sim w \Leftrightarrow \exists$ a walk from v to w in G. $\sim$ is an equivalence relation.

Proof – Reflexive $v \sim v$: take walk v. Symmetric: $v \sim w \Rightarrow \exists$ walk v to w, reverse it. Transitivity $v \sim w$ and $w \sim z$, then concatenate the $v \sim w$ and $w \sim z$ walks to give a $v \sim z$ walk.

Let $V = V_1 \cup V_2 \cup \cdots \cup V_k$ be the partition of V induced by $\sim$. We call the equivalence classes V_i <u>components</u>.

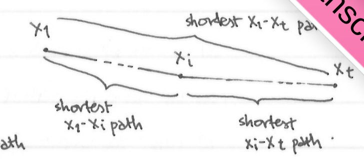
Note that by Lemma 1.11 and Lemma 1.12, $\exists$ a v-w path $\Leftrightarrow$ v and w belong to the same component in G.



three components.

We say that G is <u>connected</u> if it consists of a single component.

Lemma 1.13 let $P = x_1x_2\cdots x_t$ be a path in a graph G. If P is a shortest x_1-x_t path in G, then $x_1x_2\cdots x_i$ and $x_ix_{i+1}\cdots x_t$ are the shortest x_1-x_i and x_i-x_t paths in G for each $1<i<t$.



Proof – Assume that $\exists$ a shorter x_1-x_i path than the one specified. Then, following that path from x_1 to x_i, and then P to x_t, this x_1-x_t path is shorter than P $\Rightarrow$ contradiction $_{/\!/}$ q.e.d. Same argument for x_i-x_t part.

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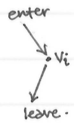
1.7 Euler circuits.

An Euler circuit in a graph G is a $\overset{\text{start=end}}{\text{closed}}$ $\overset{\text{no repeated edges.}}{\text{tour}}$ $v_0v_1\cdots v_tv_0$ containing all vertices and edges of G, the vertices may be repeated but each edge is used exactly once.

Theorem 1.14 (Euler, 1735)

A graph G has an Euler circuit iff G is connected and all vertices have even degree.

Proof – ($\Rightarrow$) G has an Euler circuit T. So G is certainly connected. let $T = v_0v_1\cdots v_k$, $\overset{v_0=v_k.}{}$ Follow T counting the contribution to the degree of each vertex we visit. Add 2 each time for entry and exit, except at start and end. Hence, all degrees are even.

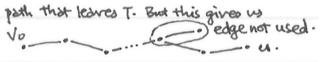


($\Leftarrow$). Suppose G is connected and all vertices have even degree. Take a longest tour $T = v_0v_1\cdots v_k$, in G. We claim that $v_0 = v_k$. If not, let j be $j = \#\{i : v_i = v_k\}$ (i.e. number of times v_k is visited). If $v_0 \neq v_k$, we have used $2(j-1)+1 = 2j-1$ edges incident to v_k. Since v_k has even degree $\exists$ an unused edge $v_kv^* \Rightarrow T' = v_0v_1\cdots v_kv^*$ is a longer tour, which is a contradiction. Hence $v_0 = v_k$.

If there is an unused edge, say $e = uv$, there are 2 cases to consider:

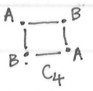
Case I: u or v is in tour, say $v = v_i$. Take $T' = uv_iv_{i+1}\cdots \overset{(v_k)}{v_0}v_1\cdots v_{i+1}$. Then T' is a longer tour than T.

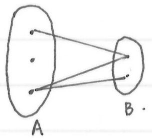
Case II: u and v are not in tour. G is connected, so $\exists$ a v_0-u path. consider the first edge in this path that leaves T. But this gives us case I $\Rightarrow$ contradiction $_{/\!/}$ q.e.d.



1.8 Bipartite graphs.

Recall that a graph G is bipartite if $V(G) = A \dot{\cup} B$ and $E(G) \subseteq \{ab : a \in A, b \in B\}$. We say that A, B is a bipartition: expressing this by $G = (A, B; E)$. Bipartitions are not necessarily unique: just defined such that there are no edges within each bipartition. We can extend this theory to tripartite etc. graphs. Smallest graph that is not bipartite is C_3. C_4 is bipartite.





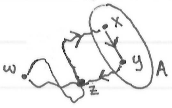
Example of an incomplete bipartite graph.

Theorem 1.15 A graph is bipartite $\Leftrightarrow$ it contains no odd cycle.

Proof – ($\Rightarrow$) Suppose G is bipartite with bipartition $V = A \dot{\cup} B$. If $C = v_1\cdots v_t$ is a cycle in G and WLOG $v_1 \in A$, then $v_3, v_5, \cdots \in A$; $v_2, v_4, \cdots \in B$. Hence, t must be even $_{/\!/}$ q.e.d.

($\Leftarrow$) Suppose G is connected (otherwise, if it is not connected, repeat this argument for each connected component). Hence, lengths between vertices is defined. For $x, y \in V$, let $d(x,y)$ = length of a shortest x-y path. Fix a vertex $w \in V$. Define $A = \{v : d(v,w) \text{ is odd}\}$, $B = \{v : d(v,w) \text{ is even}\}$. Then $V(G) = A \dot{\cup} B$. We need to check that A and B do not contain edges. By contradiction, Suppose $\exists$ edge xy inside A i.e. $x, y \in A$

let P_{wx} be a shortest w-x path, P_{wy} be a shortest w-y path. Let z be the last common vertex of P_{wx} and P_{wy}. Then the part of P_{wx} from w to z is the shortest w-z path; the part of P_{wy} from w to y is the shortest w-z as well. Then both have length $d = d(w,z)$. Suppose $d(w,x) = 2i+1$, $d(w,y) = 2j+1$. Then the cycle that begins as z, follows P_{wx} to x, takes edge xy and follows $-P_{wy}$ from y to z; has length $[(2i+1)-d]+1+[(2j+1)-d] = 2(i+j-d)+1$, which is odd $\Rightarrow$ odd cycle $\Rightarrow$ contradiction Hence, no edges inside A (and by association B) $\Rightarrow$ G is bipartite $_{/\!/}$ q.e.d.



we do not know if the paths intersect.

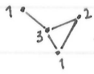
1.9 Graph colouring.

Question: what is the minimum number of colours needed to colour vertices s.t. adjacent ones have different colours?

A set $A \subset V$ is independent iff it contains no edges. For $k \in \mathbb{N}$, a k-colouring of a graph G is $V(G) \rightarrow [k]$ s.t. $vw \in E \Rightarrow c(v) \neq c(w)$.

A graph G is said to be k-colourable iff it has a k-colouring i.e. bipartite graph $\Leftrightarrow$ it is 2-colourable. A graph is k-partite if $V(G) = \dot{\bigcup}_{i=1}^{k} V_i$, where V_i are independent sets. G is k-partite $\Leftrightarrow$ G is k-colourable (different ways of looking at the same thing).

Definition The chromatic number of G, $\chi(G)$ is defined s.t. $\chi(G) = \min\{k \geq 1 : G \text{ is } k\text{-colourable}\}$.



$\chi(G) = 3$.

Note that $\chi(K_t)$, $\chi(C_{2t})=2$ and $\chi(C_{2t+1})=3$.

If H is a subgraph of G, then $\chi(H) \leq \chi(G)$, using the same colouring scheme.

$\boxed{\text{Theorem}}$ 1.16 (Greedy Algorithm of colouring)

If G is a graph, then $\chi(G) \leq \Delta(G)+1$, where $\Delta(G) = \max\{d(v) : v \in V(G)\}$.

Proof – Let $V=\{v_1, \dots, v_n\}$ be an ordered set of vertices. Let $k=\Delta(G)+1$. Define a k-colouring $c: V(G) \to [k]$ as follows:

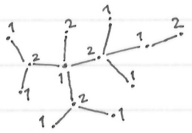
Take $c(v_1)=1$. If $v_1, \dots, v_{i-1}$ have been coloured, let $C=\{c \in [k] : \exists j \in [i-1] \text{ s.t. } v_j \in \Gamma(v_i) \text{ and } c(v_j)=c\}$ be the set of "forbidden colours".
v_j is a neighbour of v_i

Define $c(v_i) = \min [k] \setminus C$, which is well-defined by the well-ordering property, provided $[k]\setminus C$ is non-empty. $|C| \leq d(v_i) \leq \Delta(G) = k-1$, and $[k]\setminus C \neq \phi$ // q.e.d.

1.10 Large girth and large chromatic number.

This is a more modern topic in graph theory. If we start somewhere and go for a walk aiming to get back to the same point, what is the shortest length of a walk?

If G is a graph, then the girth of G, $g(G)$, is the length of the shortest cycle. If G contains no cycles, we define $g(G)=\infty$.



a graph with large girth only have cycles of long length; but shouldn't that mean that we only need few colours?

$\boxed{\text{Theorem}}$ 1.17 (Erdős, 1959)

For $k, l \geqslant 3$, $\exists$ graph G with $\chi(G) \geqslant k$, $g(G) \geqslant l$.

Note: We prove this probabilistically, first using some lemmata... It will require quite a lot of background first!

We define the independence number of G, $\alpha(G) = \max\{|A| : A \subseteq V(G) \text{ is an independent set}\}$.

$\boxed{\text{Lemma}}$ 1.18 For any graph G, $n=|V(G)|$, we have $\chi(G) \geqslant \frac{n}{\alpha(G)}$.

Proof – If $c: V(G) \to [k]$ is a k-colouring of G, then each colour class $c^{-1}(i) = \{v \in V(G) : c(v)=i\}$ is an independent set.

Hence, $|c^{-1}(i)| \leq \alpha(G)$. But $V(G) = c^{-1}(1) \dot\cup c^{-1}(2) \dot\cup \cdots \dot\cup c^{-1}(k)$; so $\sum_{i=1}^{k} |c^{-1}(i)| = n$. Hence, $|c^{-1}(i)| \leq \alpha(G) \Rightarrow k\alpha(G) \geqslant \sum_{i=1}^{k} |c^{-1}(i)| = n$

$\therefore$ $k \geqslant \frac{n}{\alpha(G)}$. Since c is a k-colouring, $\chi(G) \geqslant \frac{n}{\alpha(G)}$ // q.e.d.

uniform probability.

We consider finite, discrete probability spaces. A <u>probability space</u> is a pair (Ω, P_u) where Ω is a finite set of outcomes, $P: \Omega \to [0,1]$ e.g. for a fair die, (Ω, P_u): $\Omega=[6]$, $P_u(i)=\frac{1}{6}$, $i \in [6]$
Obviously $\sum_{y \in \Omega} P[y]=1$.

For $A \subseteq \Omega$, define $P[A] = \sum_{y \in A} P(y)$.

A <u>random variable</u> is a function $X: \Omega \to \mathbb{R}$. e.g. If our probability space is $([6], P_u)$ where $P_u(i)=\frac{1}{6}$, $i \in [6]$, we can have $\underbrace{X_1(y)=\begin{cases}1 & y=1,3,5 \\ 0 & \text{otherwise}\end{cases} \text{ or } X_2(y)=\begin{cases}1, & y \geqslant 4 \\ 0, & \text{otherwise}\end{cases}}_{\text{we call these indicator variables; as they can only take binary values 0 and 1.}}$

The <u>expectation</u> of a random variable is its average value.

If $O_X = \{X(y) \mid y \in \Omega\}$ is the set of values taken by X, then $E[X] = \sum_{z \in O_X} z P[X=z]$.

$\boxed{\text{Lemma}}$ 1.19 (Linearity of Expectation).

If $X_1, X_2, \dots, X_n$ are random variables on the same

Proof – Follows from definition of expectation.

Note: Does not require assumption of independence!

To show how we can use this idea, note that $\frac{1}{n}E\left[\sum_{i=1}^{n} X_i\right] = \mu \Rightarrow \exists X_i$ s.t. $E(X_i) \leq \mu$, and X_i s.t. $E(X_i) \geqslant \mu$. [Think: if average height in class is 5'8"; someone must be at least that height, someone at most].

$\boxed{\text{Theorem}}$ 1.20 If G is a graph with e edges, then G contains a bipartite subgraph with at least $\lceil \frac{e}{2} \rceil$ edges. (or at most $\lfloor \frac{e}{2} \rfloor$ edges).

Proof – Consider a random bipartition of $V = A \dot\cup B$. For each vertex $v \in V$, flip an independent fair coin $\Rightarrow$ if heads, put v in A; tails, put v in B.

For an edge $uv \in E$, let $X_{uv} = \begin{cases}1, & uv \text{ goes from } A \text{ to } B \\ 0, & \text{otherwise.}\end{cases}$ Let $X = \sum_{uv \in E(G)} X_{uv}$. Then $E[X] = E\left[\sum_{uv \in E(G)} X_{uv}\right] = \sum_{uv \in E(G)} E[X_{uv}] = \sum_{uv \in E(G)} P(uv \text{ goes from } A \text{ to } B)$.

$P(uv \text{ goes from } A \text{ to } B) = \frac{1}{2} \Rightarrow E[X] = \sum_{uv \in E(G)} \frac{1}{2} = \frac{e}{2}$. Thus, there must exist a bipartition with at least $\lceil \frac{e}{2} \rceil$ edges between A and B.

(Likewise exists one with at most $\lfloor \frac{e}{2} \rfloor$ edges).

Note: This is an existential proof, which merely shows that something does exist, without describing it.

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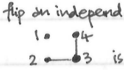
Using a similar approach, we can generate random graphs on $[n]$. We call these <u>Erdős–Renyí graphs</u>, $G(n,p)$.

$V(G)=[n]$. For each ij edge $(1 \leq i < j \leq n)$ flip an independent coin with prob(Heads) $=p$. Insert the edge ij in $E(G)$ iff the coin is Heads.

e.g. If $n=4$, and we have the graph  is labelled H, $H \in \mathcal{G}(4,p)$ (probability space). Then by Bernoulli trials, $P(G=H) = p^2(1-p)^4$
2 heads (edges) 4 tails (missing edge).

Consider a room with people of average height 5 ft. At most only half of the people can have height 10 ft – because for the other half, height must be a positive quantity. This gives us a lemma:

Lemma 1.21 (Markov's Inequality)

If X is a non-negative random variable (taking finite values in O_X), $\lambda > 0$, then $P[X \geqslant \lambda] \leqslant \frac{E[X]}{\lambda}$.

Proof – Let X take values from O_X. $E[X] = \sum_{y \in O_X} y P[X=y] \geqslant \sum_{y \geqslant \lambda} \lambda P[X=y] = \lambda \sum_{y \geqslant \lambda} P[X=y] = \lambda P[X \geqslant \lambda]$.

Earlier, we introduced Erdős-Renyi graphs; we denote their probability space as $\mathcal{G}(n,p)$. The underlying set of outcomes is $\Omega = \{G \mid V(G) = [n], E(G) \leqslant \binom{[n]}{2}\}$.

Lemma 1.22 Let $G \in \mathcal{G}(n,p)$. Let X_t be the number of t-cycles in G. Then $E[X_t] = \left(\frac{n(n-1)\cdots(n-t+1)}{2t}\right) p^t$.

Proof – Fix a t-cycle C. Let $Y_C = \begin{cases} 1, & C \text{ is in } G \\ 0, & \text{otherwise} \end{cases}$ be an indicator variable. Then $X_t = \sum_{C \text{ is a } t\text{-cycle}} Y_C \Rightarrow E[X_t] = \sum_{C \text{ is } t\text{-cycle}} E[Y_C] = \sum_{C \text{ a } t\text{-cycle}} P[C \text{ in } G]$.

But $P[C \text{ is in } G] = p^t$ for any t-cycle C. $E[X_t] = p^t \times$ # t-cycles possible in G. Any t-tuple of distinct vertices $v_1, \dots, v_t$ gives rise to a t-cycle. # such t-tuples $= n(n-1)\cdots(n-t+1)$. However, we can order them either in increasing or decreasing order of vertices $\begin{cases} v_1 v_2 \cdots\cdots v_{t-1} v_t \text{ or} \\ v_t v_{t-1} \cdots\cdots v_2 v_1 \end{cases}$; or we can start cycle from any vertex v_i, $1 \leqslant i \leqslant t \Rightarrow$ each such t-cycle coincides with $2t$ t-tuples $\Rightarrow$ # possible t-cycles $= \frac{n(n-1)\cdots(n-t+1)}{2t}$

$\therefore E[X_t] = \frac{n(n-1)\cdots(n-t+1)}{2t} p^t$ // q.e.d.

Finally, we can prove theorem 1.17:

(Theorem 1.17) Proof – Let k, l be given. We call a cycle "short" if it has length $\leqslant l$. We claim that: if $\exists$ a graph G with n vertices and at most $\frac{n}{2}$ short cycles with $\alpha(G) < \frac{n}{2k}$, then $\exists$ G' with $\chi(G') > k$ and $g(G') > l$. (We are seeking G'). Remove a vertex from each short cycle to give G'.

$|V(G')| \geqslant \frac{n}{2}$; and $g(G') > l$ as it has no short cycles left. $\alpha(G') \leqslant \alpha(G) < \frac{n}{2k}$ because independent sets of G' are subsets of independent sets of G.

Thus, $\chi(G') \geqslant \frac{|V(G')|}{\alpha(G')}$. So $\chi(G') > \frac{n}{2} \cdot \frac{2k}{n} = k$. Hence, we have proved the claim, and $\exists G \Rightarrow \exists G'$ which meets our condition.

Now, NTP: $\exists$ G with $|V(G)| = n$, at most $\frac{n}{2}$ short cycles and $\alpha(G) < \frac{n}{2k}$. Let $n \geqslant 36l^2$, $\frac{n^{\frac{1}{2l}}}{8 \log n} \geqslant 2k$. We also set probability $p = \frac{1}{n^{1-\frac{1}{2l}}}$.

We let $G \in \mathcal{G}(n,p)$ for this probability space. Let $X_t =$ # t-cycles in G. By Lemma 1.22, $E[X_t] = \frac{n(n-1)\cdots(n-t+1)}{2t} p^t$. Then $X = \sum_{t=3}^{l} X_t$ gives the # short cycles in G. Then $E[X] = \sum_{t=3}^{l} \frac{n(n-1)\cdots(n-t+1)}{2t} p^t \leqslant \sum_{t=3}^{l} \frac{n^t}{2t\, n^{t(1-\frac{1}{2l})}} \leqslant \underbrace{l}_{l \text{ terms}} \cdot \underbrace{n^{\frac{1}{2}}}_{\text{maximum}} \leqslant \frac{n}{6} \because n \geqslant 36l^2$. Then by Markov's inequality, $P(X \geqslant \frac{n}{2}) \leqslant \frac{E[X]}{n/2} \leqslant \frac{1}{3}$.

So we have $P(G \text{ has less than } \frac{n}{2} \text{ short cycles}) \geqslant \frac{2}{3}$. Next: need to show also that we have $P(\alpha(G) \geqslant \frac{n}{2k}) \leqslant \frac{1}{3}$ s.t. $P(\alpha(G) < \frac{n}{2k}) \geqslant \frac{2}{3}$.

Let B be the event "$\alpha(G) \geqslant \frac{n}{2k}$". Let $s = \frac{4}{p} \log n + 1$, then $\frac{n^{\frac{1}{2l}}}{8 \log n} \geqslant 2k \Rightarrow \frac{n}{2k} \geqslant \frac{8n \log n}{n^{1/2l}} = \frac{8}{p} \log n \geqslant s$. Then $P(B) \leqslant P(\alpha(G) \geqslant s) = P(\overset{\exists \text{ ind. set}}{\text{of size } s})$. For a set $T \subset V(G)$, $|T| = s$, let $E_T =$ "T is an ind. set". $P(B) \leqslant P(\bigcup_{T \in \binom{V}{s}} E_T) \leqslant \sum_{T \in \binom{V}{s}} P(E_T) = \binom{n}{s}(1-p)^{\binom{s}{2}} \leqslant n^s e^{-p\binom{s}{2}}$ $(\because 1-p \leqslant e^{-p}, \binom{n}{s} \leqslant n^s) = (n\, e^{-p(\frac{s-1}{2})})^s = (n\, e^{-2\log n})^s = \frac{1}{n^s} \leqslant \frac{1}{3}$ for large n.

Since $P(G \text{ has} \leqslant \frac{n}{2} \text{ short cycles}) \geqslant \frac{2}{3}$, $P(\alpha(G) < \frac{n}{2k}) \geqslant \frac{2}{3}$; they cannot be disjoint $\Rightarrow$ $(G \text{ with} \leqslant \frac{n}{2} \text{ short cycles}) \cap (G \text{ with } \alpha(G) < \frac{n}{2k}) \neq \phi$

$\Rightarrow \exists$ G s.t. G has n vertices, and most $\frac{n}{2}$ short cycles with $\alpha(G) < \frac{n}{2k}$. $\therefore \exists$ G' s.t. $\chi(G') > k$, $g(G') > l$ // q.e.d.

Chapter 2

EXTREMAL GRAPH THEORY

23 January 2013
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Maths 506.

2.1 Hamilton cycles

A Hamilton cycle in a graph G containing all the vertices in G (exactly once).

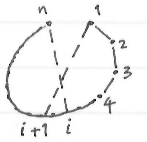
We have the minimum degree of G, $\delta(G)$, defined as $\delta(G) = \min\{d(v) : v \in V(G)\}$

Two vertices $u, v \in V(G)$ are adjacent $\Leftrightarrow$ $uv \in E(G)$. Otherwise, they are non-adjacent.

Theorem 2.1 (Dirac 1952)

If G is a graph of order $n \geqslant 3$ and $\delta(G) \geqslant \frac{n}{2}$, then G contains a Hamilton cycle.

Proof – This is an immediate corollary of the subsequent theorem.



Theorem 2.2 (Ore 1960)

If G is a graph of order $n \geqslant 3$, and $d(u) + d(v) \geqslant n$ for every pair of non-adjacent vertices, then G contains a Hamilton cycle.

Proof – By contradiction. Assume G satisfies the conditions of theorem 2.2 but does not contain a Hamilton cycle. If there is an edge that can be added to G without creating a Hamilton cycle, then do so. Repeat until no more edges can be added; getting a maximal graph. Then, any new edge would create a Hamilton cycle. So, G contains a Hamilton cycle with one edge removed. WLOG, let $V(G) = [n]$. Then $12, 23, \dots, (n-1)n \in E(G)$ by relabelling; but $1n \notin E(G)$. Note that as we begin filling up the other potential edges, we cannot have both $1(i+1)$ and $in \in E(G)$, otherwise we would have a Hamilton cycle $1(i+1)(i+2)\cdots n\, i(i-1)(i-2)\cdots 2 \cdot 1$. Consider non-adjacent vertices 1 and n. Then we evaluate $d(1) + d(n)$. Since we have at most one edge from each pair $\{13, 2n\}, \{14, 3n\}, \dots, \{1(n-1), (n-2)n\}$. $\Rightarrow$ gives $\leqslant n-3$ edges. Then, adding in the edges $12, (n-1)n \in E(G)$, we have $\deg(1) + \deg(n) \leqslant n-3+2 = n-1$ $\Rightarrow$ since $1, n$ are non-adjacent, $d(1) + d(n) \geqslant n \Rightarrow$ contradiction // q.e.d.

2.2 Forbidden subgraphs.

Given graphs G and H, we say that G is H-free if G has no subgraph isomorphic to H.

We define the extremal number, $ex(n,H) = \max\{|E(G)| : G=(V,E),\ |V|=n \text{ and } G \text{ is } H\text{-free}\}$.

Lemma 2.3 If G, H are graphs with $\chi(H) > \chi(G)$, then G is H-free.

Proof - If G contains H, then any colouring of G gives a colouring of H. Hence, $\chi(H) \leq \chi(G)$. // q.e.d.

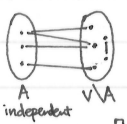
Theorem 2.4 (Mantel 1903)

If $n \geq 1$, then $ex(n,K_3) = \lfloor \frac{n^2}{4} \rfloor$.

Proof - To get triangle-free (i.e. no K_3), use a bipartite graph by lemma 2.3. To maximise $|E(G)|$, take the complete bipartite graph $K_{a,n-a}$. We seek a to maximise $|E(G)| = a(1-a)$. Take $K_{\lfloor \frac{n}{2} \rfloor, \lceil \frac{n}{2} \rceil}$. This has $\lfloor \frac{n^2}{4} \rfloor$ edges. We still need to establish that this is maximal. $ex(n,K_3) \geq \lfloor \frac{n^2}{4} \rfloor$, but NTP: $|E(G)| \leq \lfloor \frac{n^2}{4} \rfloor$.

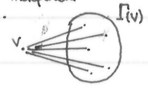
(Alternative 1).

Let $A \subseteq V(G)$ be a largest independent set in G, with $|A| = a$. We can have edges ① between A and $V \backslash A$, or ② within $V \backslash A$.



Consider $\sum_{v \in V \backslash A} d(v) \geq |E(G)|$ since we count every edge at least once (in fact we count those in $V \backslash A$ twice.

Since G is K_3-free, $\Gamma(v)$ is an independent set, for each $v \in V$. Hence, $d(v) = |\Gamma(v)| \leq a$ since no independent set is larger than a.



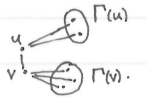
Thus, $|E(G)| \leq \sum_{v \in V \backslash A} d(v) \leq |V \backslash A| a = (n-a)a \leq \frac{n^2}{4}$ by basic calculus. Since $|E(G)| \in \mathbb{Z}$, $\lfloor \frac{n^2}{4} \rfloor$ is maximum $|E(G)|$

Hence for any graph G, $ex(n,K_3) \leq \lfloor \frac{n^2}{4} \rfloor$. We have earlier found an example, so $ex(n,K_3) = \lfloor \frac{n^2}{4} \rfloor$ // q.e.d.

(Alternative 2).

We know that $K_{\lfloor \frac{n}{2} \rfloor, \lceil \frac{n}{2} \rceil}$ be K_3-free. Then $ex(n,K_3) \geq \lfloor \frac{n^2}{4} \rfloor$. Then let G with order n be K_3-free, and set $|E(G)| = e$.

If $uv \in E(G)$, then $\Gamma(u) \cap \Gamma(v) = \phi$, since G is K_3-free. Hence, $d(u) + d(v) \leq \underbrace{(n-2)}_{u\Gamma(u),\, v\Gamma(v)} + \underbrace{2}_{uv \text{ counted twice}} = n$. Hence, $\sum_{uv \in E(G)} (d(u) + d(v)) \leq en$.



Note that if we fix a vertex x, then "d(x)" occurs once in this sum for each edge containing x, i.e. it appears d(x) times.

$\Rightarrow \sum_{uv \in E(G)} d(u) + d(v) \leq en$ means $\sum_{x \in V(G)} (d(x))^2 \leq en$. We know that $\sum_{x \in V(G)} d(x) = 2e$. By Cauchy-Schwarz inequality, $\frac{1}{n} \left(\sum_{x \in V} d(x) \right)^2 \leq \sum_{x \in V} (d(x))^2$.

Hence, $\frac{4e^2}{n} \leq \sum_{x \in V} (d(x))^2 \leq en \Rightarrow e \leq \frac{n^2}{4} \leq \lfloor \frac{n^2}{4} \rfloor$ // q.e.d.

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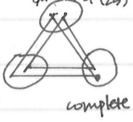
We now generalise this theory: what graphs are K_4-free? K_{r+1} free?

A graph $G = (V,E)$ is a <u>complete r-partite graph</u> if $\exists$ partition $V(G) = V_1 \dot{\cup} V_2 \dot{\cup} \cdots \dot{\cup} V_k$, where each V_i is an independent set and we have $E(G) = \{vw : v \in V_i,\ w \in V_j \text{ for some } 1 \leq i \neq j \leq k\}$.



complete 3-partite graph.

We define the Turán graph, $T_r(n)$, as the <u>complete r-partite graph</u>, with n vertices and r vertex classes as equal as possible. → |largest vertex class| ≤ |smallest vertex class| + 1.

This will maximise the size of the graph. We let $|E(T_r(n))| = t_r(n)$.

Lemma 2.5 Amongst all r-partite graphs with n vertices, $T_r(n)$ has the most edges. Moreover, $t_r(n) = t_r(n-r) + (r-1)(n-r) + \binom{r}{2}$.

Proof - Take an r-partite graph G of order n, with maximum number of edges. Suppose vertex classes are $V_1, \ldots, V_r$.

We can suppose G is complete r-partite. If $G \neq T_r(n)$, then $\exists V_i, V_j$ vertex classes with $|V_i| = a$, $|V_j| = b$ and $a \geq b+2$.

Remove a vertex v from V_i and add a vertex to V_j. Add the complete r-partite graph on these new vertex classes.

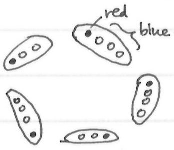
Before	a	b
After	a-1	b+1.

In doing so, we lose $(n-a)$ edges and added $(n-(b+1))$ edges ∵ edge has moved. Hence,

change in edges $= [n-(b+1)] - [n-a] = a-b-1 \geq 2-1 = +1 \Rightarrow$ this graph increases the size $\Rightarrow$ G is not maximal in size $\Rightarrow$ contradiction. ∴ $G = T_r(n)$ // q.e.d.

$\exists$ a copy of $T_r(n-r)$ inside $T_r(n)$ given by removing a vertex in each class. We colour the r vertices in $T_r(n) \backslash T_r(n-r)$ red.

Colour the rest blue. # blue-blue edges $= |E(T_r(n-r))| = t_r(n-r)$. # red-red edges $= \binom{r}{2}$, # red-blue edges $= \underbrace{(r-1)}_{\substack{\#\text{red vertices it connects to.} \\ \text{(all except the one in its own vertex class).}}} \underbrace{(n-r)}_{\#\text{blue vertices}}$



Hence, $t_r(n) = t_r(n-1) + (r-1)(n-r) + \binom{n}{2}$ // q.e.d.

Theorem 2.6 (Turán 1941).

If $2 \leq r \leq n$ and G is K_{r+1}-free of order n with $ex(n, K_{r+1})$ edges, then G is $T_r(n)$.

Proof - Induction on n. If $n \leq r$, then $ex(n, K_{r+1}) = \binom{n}{2}$ and $T_r(n) = K_n$. So suppose $n \geq r+1$. Let G have n vertices and $ex(n, K_{r+1})$ edges.

By maximality of # edges in G, then $\exists$ a copy K of K_r (otherwise we could add an edge and still be K_{r+1}-free). Let $V(K) = \{v_1, \ldots, v_r\}$.

By our inductive hypothesis, G-K has $\leq t_r(n-r)$ edges; and each $v \in V(G-K)$ has at most r-1 neighbours in V(K).

So, $|E(G)| \leq \underset{\uparrow \atop \#\text{ edges in } K}{\binom{r}{2}} + \underset{\uparrow \atop \#\text{edges in } G-K}{t_r(n-r)} + \underset{\uparrow \atop {\#\text{edges} \atop G-K \text{ to } K}}{(n-r)(r+1)} = t_r(n)$. Hence, by maximality of $|E(G)|$, equality must hold i.e. $|E(G)| = t_r(n)$. For equality to hold, each vertex $v \in V(G-K)$ must have exactly r-1 neighbours in V(K). For $1 \leq i \leq r$, let $W_i = \{v \in V(G) : vv_i \notin E(G)\}$

Then $v_i \in W_i$, $v_i \notin W_j$ for all $i \neq j$. If $v \in V(G-K)$, v has exactly r-1 neighbours in $V(K) \Rightarrow \exists$ unique $1 \leq i \leq r$ st. $vv_i \notin E(G)$, hence $v \in W_i$ for some unique i.

$\therefore\ \omega_1 \dot\cup \cdots \dot\cup \omega_r$ is a partition of $V(G)$. If $u,v \in \omega_i$ and $uv \in E(G)$, then $u, v, v_1, v_2, \dots, v_{i-1}, v_{i+1}, \dots, v_r$ forms $K_{r+1} \Rightarrow$ contradiction $\Rightarrow$ independent sets. G is an r-partite graph with vertex classes $\omega_1, \dots, \omega_r$. By Lemma 2.5, $G = T_r(n)$// q.e.d.

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Definition If $G=(V,E)$ is a graph, then the complement of G is $G^c = (V, \binom{V}{2} \setminus E)$. Hence, $G \cup G^c = K_{|V|}$.

Theorem 2.7 (Caro and Wei 1979/81)

If G is a graph of order n, with vertex degrees (degree sequence) $d_1, \dots, d_n$, then $\alpha(G) \geqslant \sum_{i=1}^{n} \frac{1}{d_i + 1}$.

In particular, if all vertices have degree d, then $\alpha(G) \geqslant \frac{n}{d+1}$.

Proof – Take $V(G) = [n]$. Choose $\pi \in S_n$ uniformly at random. Let A_i be the event that "$\pi(i) < \pi(j)$ for every $j \in \Gamma(i)$" i.e. A_i holds $\Leftrightarrow$ amongst $\{i\} \cup \Gamma(i)$, i is "first" among ordering given by π.
For each π,
let $U = \{i \in V(G) : A_i \text{ holds}\}$. Suppose $a, b \in U$, $ab \in E$. Then $a \in \Gamma(b)$ and $b \in \Gamma(a)$. But $A_a \Rightarrow \pi(a) < \pi(b)$, $A_b \Rightarrow \pi(b) < \pi(a)$.

Hence, $ab \notin E \Rightarrow U$ is an independent set. $P(A_i \text{ holds}) = P(\text{In a random ordering of } \{i\} \cup \Gamma(i), \text{ "}i\text{" comes first}) = \frac{1}{d_i+1}$.

Since U is an independent set, then $\alpha(G) \geqslant |U| \Rightarrow E[\alpha(G)] \geqslant E[|U|] \Rightarrow \alpha(G) \geqslant E|U| = \sum_{i=1}^{n} P(A_i \text{ holds}) = \sum_{i=1}^{n} \frac{1}{d_i+1}$// q.e.d.

Remark: This is another way of equivalently stating Turán's theorem.

Take $C_5^* =$  as shown. What is $ex(n, C_5^*)$? or in general $ex(n,H)$? It is difficult to tell in general.

We define the Turán density of H by $\pi(H) = \lim_{n\to\infty} \frac{ex(n,H)}{\binom{n}{2}}$. Is this a well-defined limit?

Lemma 2.8 For a graph H, $\pi(H)$ is well-defined. If $r \geqslant 2$, then $\pi(K_{r+1}) = 1 - \frac{1}{r}$.

Proof – We know that $\max ex(n,H) = \binom{n}{2}$, so $\frac{ex(n,H)}{\binom{n}{2}}$ is bounded (above by 1, below by 0). We claim $\left\{\frac{ex(n,H)}{\binom{n}{2}}\right\}_{n=1}^{\infty}$ is monotone decreasing.

Let G be H-free, with order n and $ex(n,H)$ edges. Consider $\sum_{v \in V(G)} |E(G-v)|$. Since $G-v$ has order $n-1$, $|E(G-v)| \leqslant ex(n-1,H)$ for each $v \in V \Rightarrow \sum_{v \in V(G)} |E(G-v)| \leqslant n\, ex(n-1,H)$. But $\sum_{v\in V(G)} |E(G-v)| = \underset{\text{each edge counted n-2 times}}{(n-2)} |E(G)| = (n-2)\, ex(n,H) \Rightarrow (n-2)\, ex(n,H) \leqslant n\, ex(n-1,H)$
$\Rightarrow \frac{2\, ex(n,H)}{n(n-1)} \leqslant \frac{2\, ex(n-1,H)}{(n-1)(n-2)} \Rightarrow \frac{ex(n,H)}{\binom{n}{2}} \leqslant \frac{ex(n-1,H)}{\binom{n-1}{2}} \Rightarrow$ sequence is monotone decreasing// q.e.d.

By Turán's theorem, $ex(n, K_{r+1}) = t_r(n)$, # edges in a complete r-partite graph with vertex classes of size $\lceil \frac{n}{r} \rceil$ or $\lfloor \frac{n}{r} \rfloor$.

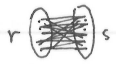
Hence $\binom{r}{2} \lfloor \frac{n}{r} \rfloor^2 \leqslant t_r(n) \leqslant \binom{r}{2} \lceil \frac{n}{r} \rceil^2$. $\Rightarrow \frac{\binom{r}{2}\left(\frac{n-r}{r}\right)^2}{\binom{n}{2}} \leqslant \frac{t_r(n)}{\binom{n}{2}} \leqslant \frac{\binom{r}{2}\left(\frac{n+r}{r}\right)^2}{\binom{n}{2}} \Rightarrow \left(\frac{r-1}{r}\right) \frac{(n-r)^2}{n(n-1)} \leqslant \frac{t_r(n)}{\binom{n}{2}} \leqslant \left(\frac{r-1}{r}\right) \frac{(n+r)^2}{n(n-1)}$.

Fix r, and take $n \to \infty$. Then $\lim_{n\to\infty} \frac{t_r(n)}{\binom{n}{2}} = \pi(K_{r+1}) = 1 - \frac{1}{r}$// q.e.d.

If $\pi(K_{r+1}) = 1 - \frac{1}{r}$, $r \geqslant 2$, surely then $\pi(K_{r+1}) \in \{0, \frac{1}{2}, \frac{2}{3}, \dots\}$. We will eventually show that Turán densities are always restricted to this set.

2.3 Bipartite forbidden subgraphs.

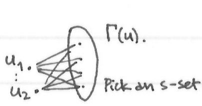
Theorem 2.9 (Kővári–Sós–Turán 1954).



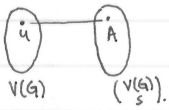
If $r, s \geqslant 2$ and n is large, $ex(n, K_{r,s}) \leqslant \frac{1}{2}(r-1)^{\frac{1}{s}} n^{2-\frac{1}{s}} + \frac{1}{2}(s-1)n$.

Proof – Let G be $K_{r,s}$-free, order n with e edges. Then if $u \in V(G)$ and $A = \{v_1, \dots, v_s\} \in \binom{V(G)}{s}$, then u covers A, if $uv_1, uv_2, \dots, uv_s \in E(G)$.

So u covers $\binom{d(u)}{s}$ s-sets. How many vertices can cover the same s-set A? Clearly since G is $K_{r,s}$-free, at most r-1 vertices can cover the same s set. Form a bipartite graph H, introduce an edge from $u \in V(G)$ to $A \in \binom{V(G)}{s} \Leftrightarrow$ u covers A. Now, we count the number of edges in H: $|E(H)| = \sum_{u \in V(G)} d_H(u) = \sum_{u \in V(G)} \binom{d_G(u)}{s}$.



H:

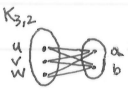


Simultaneously, $|E(H)| = \sum_{A \in \binom{V(G)}{s}} d_H(A) \leqslant \sum_{A \in \binom{V(G)}{s}} (r-1)$. Thus, $\sum_{u \in V(G)} \binom{d(u)}{s} \leqslant (r-1)\binom{n}{s}$. We know $\sum_{u \in V(G)} d(u) = 2e$. By convexity of binomial coefficients and Jensen's inequality, we get $\binom{2e/n}{s} n \leqslant (r-1)\binom{n}{s}$. Let $\alpha \geqslant 0$ be defined by $e = n^{2-\alpha}$. $\Rightarrow n\binom{2n^{1-\alpha}}{s} \leqslant (r-1)\binom{n}{s}$.

Recall that $\frac{(a-b+1)^b}{b!} \leqslant \binom{a}{b} \leqslant \frac{a^b}{b!}$; so we have $n(2n^{1-\alpha} - s + 1)^s \leqslant (r-1)n^s \Rightarrow 2n^{1-\alpha} - s + 1 \leqslant (r-1)^{\frac{1}{s}} n^{1-\frac{1}{s}}$. As such,
$e = n^{2-\alpha} \leqslant \frac{1}{2}(r-1)^{\frac{1}{s}} n^{2-\frac{1}{s}} + \frac{(s-1)}{2} n$// q.e.d.

Corollary 2.10 (Erdős 1946)

Let $X \subseteq \mathbb{R}^2$, $|X| = n$. Then at most $\frac{n^{3/2}}{\sqrt{2}} + \frac{n}{2}$ pairs of points in X are at unit distance.

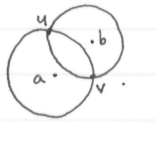


Proof – Consider the graph formed by pairs of points at unit distance. Claim that this graph is $K_{3,2}$ free.

Let a, b be at unit distance, and form two circles of unit radius around them.

Both u, v are at unit distance from both a and b $\Rightarrow$ they are at intersections $\Rightarrow$ we cannot place w on graph.

$\Rightarrow$ Graph is $K_{3,2}$ free as two unit circles meet at most twice on $\mathbb{R}^2$. So # pairs of points at unit distance $= |E(G)|$,
$|E(G)| \leqslant ex(n, K_{3,2}) = \frac{1}{2}(3-1)^{\frac{1}{2}} n^{2-\frac{1}{2}} + \frac{1}{2}(2-1)n = \frac{\sqrt{2}}{2} n^{\frac{3}{2}} + \frac{1}{2}n$// q.e.d.



2.4 The fundamental theorem of extremal graph theory.

We now move on to a central theorem of this chapter:

Theorem 2.11 (Erdős-Stone 1946)

If $\chi(H)=r$, then $\pi(H)=1-\frac{1}{r-1}$.

Proof – We want to show both that $\pi(H)\leq 1-\frac{1}{r-1}$ and $\pi(H)\geqslant 1-\frac{1}{r-1}$. Let H be given. Suppose $\chi(H)=r\geqslant 2$. $\Rightarrow$ H is r-partite, so $Tr_{-1}(n)$ is H-free $\Rightarrow ex(n,H)\geqslant |E(T_{r-1}(n))| = t_{r-1}(n)$. Then $\frac{ex(n,H)}{\binom{n}{2}} \geqslant \frac{t_{r-1}(n)}{\binom{n}{2}} \rightarrow 1-\frac{1}{r-1}$. Then $\pi(H)\geqslant 1-\frac{1}{r-1}$.

Let $K_r(t)$ be the complete r-partite graph with t vertices in each class (it has rt vertices). If $t\geqslant |V(H)|$, then $K_r(t)$ contains a copy of H. Hence, $\pi(H)\leq \pi(K_r(t))$. So it is sufficient to prove that $\pi(K_r(t))\leq 1-\frac{1}{r-1}$. We will continue this after proving some preliminary results.

First we will show how to convert conditions on the number of edges in a graph into information about minimum degrees.

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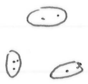
Lemma 2.12 Let $0<c,\varepsilon<1$ and $n>\frac{2}{\varepsilon}(1+\frac{1}{c})$. If G is a graph of order n and at least $(c+\varepsilon)\binom{n}{2}$ edges, then G contains a subgraph G', of order $n'\geqslant \varepsilon^{1/2}n$, with minimum degree $\delta(G')\geqslant cn'$.

Theorem 2.13 Let $r,t\geqslant 1$ and $0<\varepsilon<\frac{1}{r}$. Then $\exists\, n_0(r,t,\varepsilon)$ s.t. if G has $n\geqslant n_0$ vertices and $\delta(G)\geqslant (1-\frac{1}{r+1}+\varepsilon)n$, then G contains a copy of $K_r(t)$.

Scheme of proofs: We prove Theorem 2.13, and assuming Lemma 2.12, we convert the proof into a proof for Theorem 2.11.

(Theorem 2.11) Proof cont'd – We know $T_{r-1}(n)$ is H-free, so $\pi(H)\geqslant \pi(K_r)=1-\frac{1}{r-1}$. Also if $t\geqslant |V(H)|$ then $H\subseteq K_r(t)$. $\chi(H)=r \Rightarrow$ H =



r-partite graph.

Then $K_r(t)$ contains H. $\Rightarrow \pi(H)\leq \pi(K_r(t))$. Need to show: $\pi(K_r(t))\leq 1-\frac{1}{r-1}$.

Suppose this fails to hold, then $\exists\, \varepsilon>0$ s.t. $\pi(K_r(t)) > 1-\frac{1}{r-1}+3\varepsilon$. Let $n\geqslant \frac{n_0(r,t,\varepsilon)}{\varepsilon^{1/2}}$ given by Theorem 2.13, and let G be $K_r(t)$-free graph of order n and at least $(1-\frac{1}{r}+2\varepsilon)\binom{n}{2}$ edges. By lemma 2.12, with $c=1-\frac{1}{r-1}+\varepsilon$, G contains a subgraph G' of order $n'\geqslant \varepsilon^{\frac{1}{2}}n\geqslant n_0(r,t,\varepsilon) \Rightarrow \delta(G')\geqslant (1-\frac{1}{r-1}+\varepsilon)n'$. So Theorem 2.13 $\Rightarrow K_r(t)\subseteq G' \Rightarrow$ contradiction since $G'\subseteq G$ is $K_r(t)$ free.

(Lemma 2.12) Proof – We find G' as follows. Let $G_n=G$, $|V(G_n)|=n$. If $\delta(G_n)\geqslant cn$, then let $G'=G_n$. Otherwise, $\delta(G_n)<c_n$. Remove a vertex of minimum degree to give G_{n-1}. If $\delta(G_{n-1})\geqslant c(n-1)$, then $G'=G_{n-1}$. Otherwise, continue this algorithm... Repeat until we construct a sequence of graphs $G_n, G_{n-1}, G_{n-2},\ldots$ where $|V(G_k)|=k$ and we obtain G_{k-1} from G_k by deleting a vertex of minimum degree.

We claim this process terminates at some $k\geqslant \varepsilon^{1/2}n$. Otherwise, if $s=\lceil \varepsilon^{1/2}n\rceil$ then $|E(G_s)| > |E(G)| - \sum_{k=s+1}^{n} ck$ ← maximum possible no. of edges lost by assuming condition.

$|E(G_s)| > |E(G)| - \sum_{k=s+1}^{n} ck \geqslant (c+\varepsilon)\binom{n}{2} - c\left(\binom{n+1}{2}-\binom{s+1}{2}\right) = c\binom{n}{2}+\varepsilon\binom{n}{2}-c\binom{n}{2}-cn+c\binom{s+1}{2} = \varepsilon\binom{n}{2}-cn+c\binom{s+1}{2}$. By our choice of $s=\lceil\varepsilon^{1/2}n\rceil$, and $n>\frac{2}{\varepsilon}(1+\frac{1}{c})$. $\Rightarrow |E(G_s)|$ is evaluated using inequalities: $\binom{s+1}{2} > \frac{s^2}{2} \geqslant \frac{\varepsilon^{1/2}n}{2} > (1+\frac{1}{c})n = n+\frac{n}{c}$. Hence $|E(G_s)| > \varepsilon\binom{n}{2}+n$

so $\varepsilon\binom{n}{2}+n \leq \binom{s}{2} \leq \frac{(\varepsilon^{1/2}+1)(\varepsilon^{1/2})}{2} \Rightarrow \varepsilon n^2-\varepsilon n+2n \leq \varepsilon n^2+\varepsilon^{1/2}n \Rightarrow 2\leq \varepsilon^{1/2}+\varepsilon<2 \Rightarrow$ contradiction, q.e.d.

(Theorem 2.13) Proof – By induction on r. $r=1$ is meaningless and trivial, so we start with $r=2$. $K_2(t)=K_{t,t}$.

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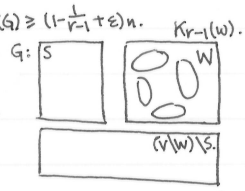
So $ex(n,K_2(t)) \leq \frac{1}{2}(t-1)^{\frac{1}{t}}n^{2-\frac{1}{t}}+\frac{1}{2}(t-1)n$ (from Kovari-Sos-Turan theorem) $< tn^{2-\frac{1}{t}}$ (crude bound).

Given $\varepsilon>0$ and $t\geqslant 1$, define $n_0(2,t,\varepsilon)$ so that for $n\geqslant n_0$, we have $\varepsilon>\frac{2t}{n^{1/t}}$. Let G be a graph with $n\geqslant n_0$ vertices and $\delta(G)\geqslant \varepsilon n$. Then G has at least $\frac{\varepsilon n^2}{2}$ edges. $\frac{\varepsilon n^2}{2} > \frac{2t}{n^{1/t}}\cdot\frac{n^2}{2} = tn^{2-\frac{1}{t}}$. Hence, $|E(G)| > ex(n,K_2(t))$ and G contains $K_2(t)$.

Now suppose $r\geqslant 3$, $t\geqslant 1$ and $0<\varepsilon<\frac{1}{r}$ is given, and the result holds for $r-1$. Let G have n vertices, $\delta(G)\geqslant (1-\frac{1}{r-1}+\varepsilon)n$.



NTP: For n sufficiently large, G contains $K_r(t)$. We construct G, with vertex set V and subsets S, W as shown.

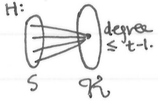
Let $w=\lceil\frac{2t}{\varepsilon}\rceil$, and let $n\geqslant n_0(r-1,w,\varepsilon)$. Since $\delta(G)\geqslant (1-\frac{1}{r-1}+\varepsilon)n > (1-\frac{1}{r-2}+\varepsilon)n$. We know G contains a copy of $K_{r-1}(w)$, with vertex set W. Then $|W|=(r-1)w$.

Let $S=\{v\in V\backslash W: v$ has $\geqslant (r-2)w+t$ neighbours inside $W\}$. Note that if $v\in S$, then v has $\geqslant t$ neighbours in each vertex class of W, so v is adjacent to all the vertices of a copy of $K_{r-1}(t)$. We claim that $|S|\rightarrow\infty$ as $n\rightarrow\infty$. In particular, if n is sufficiently large, $|S| > (t-1)\binom{w}{t}^{r-1}$. ⊛ We call a vertex $v\in S$ "good" for a copy $\hat{K}$ of $K_{r-1}(t)$ in W if v is adjacent to every vertex in $\hat{K}$.

If G is $K_r(t)$-free, then for each copy of $K_{r-1}(t)$ in W, there are at most $(t-1)$ good vertices in S.

By definition of S, every vertex in S is good for at least one copy of $K_{r-1}(t)$ in W. How many copies of $K_{r-1}(t)$ are there in W?

We pick t vertices from w in each of $r-1$ classes, so there are $\binom{w}{t}^{r-1}$ copies of $K_{r-1}(t)$. So we have the following bipartite graph H:



with components S, $\mathfrak{K}=\{\hat{K}:\hat{K}$ is a copy of $K_{r-1}(t)$ in $W\}$. $v\in S$ is joined by an edge in H to $\hat{K}\in\mathfrak{K}$ $\Leftrightarrow$ v is good for $\hat{K}$. Then $\forall\,\hat{K}\in\mathfrak{K}$, $d(\hat{K})\leq t-1$. $\forall\, v\in S$, $d(v)\geqslant 1$. Then we have

$|S|\leq \sum_{v\in S} d_H(v) = |E(H)| = \sum_{\hat{K}\in\mathfrak{K}} d_H(\hat{K}) \leq (t-1)\binom{w}{t}^{r-1}$. Then $|S|\leq (t-1)\binom{w}{t}^{r-1}$, which contradicts ⊛. Then if we can prove ⊛, we are done.

Let $e(W,V\backslash W)$ be # edges from W to its complement, $V\backslash W$. Then $\delta(G)\geqslant (1-\frac{1}{r-1}+\varepsilon)n$. there are at most $|W|^2/2$ edges inside W. ~~Also, each vertex v∈W has at most |W| neighbours~~ in W. $e(W,V\backslash W) = \sum_{v\in W} d(v) - 2e(W) \geqslant |W|n(1-\frac{1}{r-1}+\varepsilon)-|W|^2$. Recall $S=\{v\in V\backslash W: r$ has $\geqslant (r-2)w+t$ neighbours in $W\}$.

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if $v \in (V \backslash W) \backslash S$, then v has $< (r-2)w+t$ neighbours in W; if $v \in S$, then v has $<|W|$ neighbours in W.

$e(W, V \backslash W) < (\overbrace{(r-2)w+t}^{|W|-(w-t)})(n-|W|-|S|) + |S||W|$. $|W| = (r-1)w$, so we get

$e(W, V\backslash W) < n((r-2)w+t) - |W|^2 + |W|(w-t) - |S||W| + |S||W| + |S|(w-t)$. Thus, considering lower and upper bounds,

$|W|n(1-\frac{1}{r-1}+\varepsilon) - |W|^2 < n((r-2)w+t) - |W|^2 + |S|(w-t) + |W|(w-t)$. Then, $wn(r-2+(r-1)\varepsilon) < n((r-2)w+t) + |S|(w-t) + w(r-1)(w-t)$.

$\Rightarrow$ By rearrangement, $|S| > n\left(\frac{\varepsilon(r-1)w-t}{w-t}\right) - (r-1)w$. since $r \geqslant 3$, $w \geqslant \frac{2t}{\varepsilon}$, $\frac{\varepsilon(r-1)w-t}{w-t} > 0$.

Thus, as $n \to \infty$, $|S| \to \infty$ // q.e.d.

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2.5 Stability.

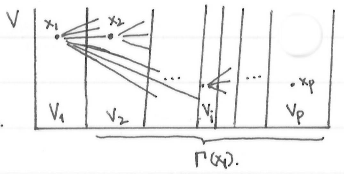
We recall Turán's results: $ex(n,H) = \max\{|E| : G=(V,E), |V|=n, G \text{ is } H\text{-free}\}$.

1) Turán's theorem: $ex(n, K_{r+1}) = t_r(n)$ 2) $\pi(H) = \lim_{n\to\infty} \frac{ex(n,H)}{\binom{n}{2}}$ exists, $\chi(H) = r \geqslant 2 \Rightarrow \pi(H) = 1 - \frac{1}{r-1}$ (Erdős-Stone) 3) Stability.

If a K_3-free graph of order n has "almost" $ex(n,K_3) = \lfloor \frac{n^2}{4} \rfloor$ edges, must it look like $T_2(n)$?

theorem 2.14 (Füredi 2010)

If G is K_{r+1}-free, order n with at least $ex(n, K_{r+1}) - t$ edges for some $t \geqslant 0$, then $\exists H \subseteq G$ s.t. $|E(H)| \geqslant |E(G)| - t$ and $\chi(H) = r$.



Proof – let $G=(V,E)$ be K_{r+1}-free, $|V|=n$ and $|E| = ex(n, K_{r+1}) - t$. choose $x_1 \in V$ of max degree.

let $V_1 = V \backslash \Gamma(x_1)$. Now consider the graph $G_2 = G[V \backslash V_1]$. chose x_2 of max degree.

let $V_2 = V(G_2) \backslash \Gamma_{G_2}(x_2)$. Repeat until we have no vertices left. suppose we chose $x_1, \dots, x_p$.

Then $x_1, \dots, x_p$ form a clique (i.e. a copy of K_p). Hence, $p \leq r$.

let $d_1 = d(x_1)$, $d_2 = d_{G_2}(x_2)$ etc. to give $d_1, \dots, d_p$. then $d_i = |V_{i+1}| + |V_{i+2}| + \dots + |V_p|$. Now for $v \in V_i$,

define $\overrightarrow{d}(v) = \#\{w : vw \in E, w \in V_i \cup V_{i+1} \cup \dots \cup V_p\}$ as the "forward degree". If $v \in V_i$, $\overrightarrow{d(v)} \leq d_i$, by

maximality of degree of x_i in G_i). then $\sum_{i=1}^{p} \sum_{v \in V_i} \overrightarrow{d(v)} = \#$ edges in G + # edges inside classes (we add this since such edges are double-counted in our summation).

Then $|E(G)| + \#^{\text{edges inside}}_{\text{classes}} = \sum_{i=1}^{p}\sum_{v\in V_i} \overrightarrow{d}(v) \leq \sum_{i=1}^{p} d_i |V_i| = \sum_{i=1}^{p} \underbrace{|V_i|(|V_{i+1}| + \dots + |V_p|)}_{\text{no. of forward edges from } V_i \text{ to beyond}} = |E(K(V_1, V_2, \dots, V_p))|$,

where $K(V_1, \dots, V_p)$ is the complete p-partite graph with vertex classes:

$V_1, V_2, \dots, V_p$. then by Lemma 2.5, $Tr(n)$ maximises edges amongst all r-partite graphs $\Rightarrow |E(G)| + \#^{\text{edges inside}}_{\text{graph}} \leq t_p(n) \leq t_r(n)$;

since $p \leq r$. Also, $|E(G)| \geqslant ex(n, K_{r+1}) - t = t_r(n) - t$ by Turán's theorem. So, we put this together to get:

edges inside class $\leq t$. let H be G with all edges inside classes removed. then $|E(H)| \geqslant |E(G)| - t$ and $H \leq K(V_1, \dots, V_p)$ is p-partite // q.e.d

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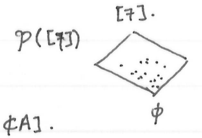
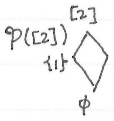
chapter 3.
SET SYSTEMS.

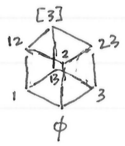
If $[n] = \{1, 2, \dots, n\}$, we have the power set of $[n]$: $\mathcal{P}([n]) = \{A : A \subseteq [n]\}$.

We note that the family of k-subsets of $[n]$ is given by $\binom{[n]}{k} = \{A \subseteq [n] : |A| = k\}$.

3.1 chains and Antichains.







We say that a family $\mathcal{A} \subseteq \mathcal{P}([n])$; if $\forall A, B \in \mathcal{A}$, $A \subseteq B$ or $B \subseteq A$; is a chain.

It is an <u>antichain</u> iff $\forall A, B \in \mathcal{A}$, $A \subseteq B \Rightarrow A = B$. [or $\forall A \neq B$, $A, B \in \mathcal{A} \Rightarrow A \not\subseteq B$ and $B \not\subseteq A$].

e.g. antichains are $\binom{[7]}{3}$, $\binom{[n]}{k}$ {1 2 3, 4 5, 1 2 4 7}

$\boxed{\text{Lemma}}$ 3.1 If $\mathcal{A}$ is an antichain and $\mathcal{C}$ is a chain, then $|\mathcal{A} \cap \mathcal{C}| \leq 1$.

Proof – if $|\mathcal{A} \cap \mathcal{C}| \geqslant 2$, let $A, B \in \mathcal{A} \cap \mathcal{C}$, $A \neq B$. Then $A, B \in \mathcal{C}$ is a chain $\Rightarrow$ wlog, $A \subset B$. But $A, B \in \mathcal{A}$ is an antichain.

Hence $A = B \Rightarrow$ contradiction $\Rightarrow |\mathcal{A} \cap \mathcal{C}| \leq 1$ // q.e.d.

$\boxed{\text{lemma}}$ 3.2 If $\mathcal{C} \subseteq \mathcal{P}([n])$ is a chain, then $|\mathcal{C}| \leq n+1$.

Proof – if $A, B \in \mathcal{C}$ and $|A| = |B|$ then $A = B$ (otherwise $\mathcal{C}$ is not a chain). Hence we have $\leq$ one set of each possible size from $\mathcal{P}([n])$.

$\therefore |\mathcal{C}| \leq n+1$ // q.e.d.

This gives us a guideline as to how large a chain $\mathcal{C} \subseteq \mathcal{P}([n])$ can be.

We know furthermore that we can partition $\mathcal{P}([n])$ into $n+1$ antichains: $\mathcal{P}([n]) = \binom{[n]}{0} \dot\cup \binom{[n]}{1} \dot\cup \dots \dot\cup \binom{[n]}{n}$.

since $\mathcal{C}$ contains at most one set from each of those antichains, $|\mathcal{C}| \leq n+1$. This is an alternate proof of lemma 3.2.

We observe that $\left|\binom{[n]}{\lfloor n/2 \rfloor}\right| = \binom{n}{\lfloor n/2 \rfloor}$, which is the largest of the binomial coefficients raised to power n. Then we get

Theorem 3.3 (Sperner 1928)

If $\mathcal{A} \subseteq \mathcal{P}([n])$ is an antichain, then $|\mathcal{A}| \leq \binom{n}{\lfloor n/2 \rfloor}$.

Lemma 3.4^* If $n \geqslant 1$ then $\mathcal{P}([n])$ can be partitioned into $\binom{n}{\lfloor n/2 \rfloor}$ chains.

Note: Lemma 3.4^* together with Lemma 3.1 $\Rightarrow$ theorem 3.3. This gives us a scheme of proof.

We first make a definition: let chain $\mathcal{C} \subseteq \mathcal{P}([n])$. Then $\mathcal{C}$ is <u>symmetric</u> if $\mathcal{C} = \{C_1, \dots, C_k\}$ with (i) $|\mathcal{C}_{i+1}| = |\mathcal{C}_i| + 1$, $i = 1, \dots, k-1$; and (ii) $|C_1| + |C_k| = n$.

For instance in $\mathcal{P}([3])$, $\{\phi, 1, 12, 123\}$, $\{2, 23\}$ are symmetric chains. In $\mathcal{P}([4])$, $\{1, 12, 124\}$ is a symmetric chain.

since $|C_1| + |C_k| = n$, we know that $|C_1| \leq \frac{n}{2}$, $|C_k| \geqslant \frac{n}{2}$. i.e. a symmetric chain $\mathcal{C} \subseteq \mathcal{P}([n])$ meets "the" middle layer $\binom{[n]}{\lfloor n/2 \rfloor}$.

since $\binom{[n]}{\lfloor n/2 \rfloor}$ is itself an antichain, we know that any symmetric chain contains exactly one set from $\binom{[n]}{\lfloor n/2 \rfloor}$, by Lemma 3.1.

Thus, we can modify Lemma 3.4^* into an equivalent form:

Lemma 3.4 If $n \geqslant 1$, then $\mathcal{P}([n])$ can be partitioned into symmetric chains (and any such partition contains exactly $\binom{n}{\lfloor n/2 \rfloor}$ chains.

Proof – Induction on n: $n=1$, $\mathcal{P}([1]) = \{\phi, 1\}$ is a symmetric chain. Now suppose $n \geqslant 2$, and result holds for $n-1$.

so $\exists$ a partition of $\mathcal{P}([n-1])$ into symmetric chains; i.e. $\mathcal{P}([n-1]) = \mathcal{C}_1 \dot\cup \mathcal{C}_2 \dot\cup \cdots \dot\cup \mathcal{C}_t$, $\mathcal{C}_i = \{C_1^i, C_2^i, \dots, C_k^i\}$

We form two new chains from $\mathcal{C}_i$: $\mathcal{C}_i' = \{C_1^i \cup \{n\}, C_2^i \cup \{n\}, \dots, C_{k_i-1}^i \cup \{n\}\}$ (if $k_i \geqslant 2$); and also

$\mathcal{C}_i'' = \{C_1^i, C_2^i, \dots, C_{k_i}^i, C_{k_i}^i \cup \{n\}\}$. Note that $\mathcal{C}_i', \mathcal{C}_i''$ are both symmetric chains (by considering orders of first/last terms) in $\mathcal{P}([n])$

Moreover, $\mathcal{P}([n]) = (\mathcal{C}_1' \dot\cup \mathcal{C}_1'') \dot\cup (\mathcal{C}_2' \dot\cup \mathcal{C}_2'') \dot\cup \cdots \dot\cup (\mathcal{C}_n' \dot\cup \mathcal{C}_n'')$ $\Rightarrow$ the result holds // q.e.d.

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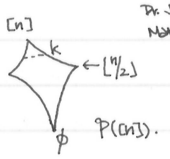
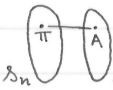


Theorem 3.5 (Lubell-Yamamoto-Meshalkin 1954)

If $\mathcal{A} \subseteq \mathcal{P}([n])$ is an antichain, then $\sum_{A \in \mathcal{A}} \frac{1}{\binom{n}{|A|}} \leq 1$.

Note: since $\binom{n}{\lfloor n/2 \rfloor} \geqslant \binom{n}{k}$ for any $0 \leq k \leq n$, the LYM-inequality $\Rightarrow$ Sperner's theorem

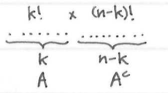
Proof – let $\mathcal{A} \subseteq \mathcal{P}([n])$ be an antichain. S_n = permutations of $[n]$. Construct a bipartite graph $G = (S_n, A; E)$.



Let $\pi \in S_n$ be joined by an edge to $A \in \mathcal{A}$ $\Leftrightarrow$ all the elements of A appear before all the elements of A^c in π.

For instance, if $n=8$, $A = 1\,3\,4$, $\pi = 1\,3\,4\,5\,6\,8\,7\,2$, then πA is an edge. Likewise if $n=7$, $A = 2\,3\,7$, $\pi = 7\,2\,3\,4\,6\,5\,1$, πA is an edge.

but if $B = 2\,3\,6\,7$ then πB is not an edge. We then employ the principle of double counting: $\sum_{\pi \in S_n} d(\pi) = |E| = \sum_{A \in \mathcal{A}} d(A)$.



If $A \in \mathcal{A}$ and $|A| = k$, then $d(A) = k!(n-k)!$. Hence $|E| = \sum_{A \in \mathcal{A}} |A|!\,(n-|A|)!$

Now if $\pi \in S_n$ and $\pi A, \pi B$ are distinct edges, then either $A \subset B$ or $B \subset A$, so $A = B$. $\therefore$ at most one edge from π $\therefore d(\pi) \leq 1$.

So $|E| = \sum_{\pi \in S_n} d(\pi) \leq \sum_{\pi \in S_n} 1 = n!$ so $\sum_{A \in \mathcal{A}} \frac{|A|!\,(n-|A|)!}{n!} \leq 1$ $\Rightarrow$ $\sum_{A \in \mathcal{A}} \frac{1}{\binom{n}{|A|}} \leq 1$ // q.e.d.

3.2 Intersecting families

A family of sets $\mathcal{A}$ is <u>intersecting</u> $\Leftrightarrow$ $A, B \in \mathcal{A} \Rightarrow A \cap B \neq \phi$. e.g. $\{12, 13, 23\}$.

Theorem 3.6 If $\mathcal{A} \subseteq \mathcal{P}([n])$ is intersecting then $|\mathcal{A}| \leq 2^{n-1}$.

Proof – since $A \in \mathcal{A} \Rightarrow A^c \notin \mathcal{A}$, hence $|\mathcal{A}| \leq 2^{n-1}$ // q.e.d.

Examples: let $\mathcal{A}^* = \{A \subseteq [n] : 1 \in A\}$, $|\mathcal{A}^*| = 2^{n-1}$. $\mathcal{B} = \{B \subseteq [n] : |B \cap [3]| \geqslant 2\}$, $|\mathcal{B}| = 4 \times 2^{n-3} = 2^{n-1}$. Then $\mathcal{B}$ consists of $B = \hat{B} \cup B'$, where

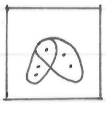
$\hat{B} \in \{12, 13, 23, 123\}$, $B' \in \{4, 5, \dots, n\}$. Set $\mathcal{C} = \{C \subseteq [n] : |C \cap [5]| \geqslant 3\}$. If $C \in \mathcal{C}$ then $C = \hat{C} \cup C'$, where

$\hat{C} \in \{123, 124, 125, 134, 135, 145, 234, 235, 245, 345, 1234, 1235, 1245, 1345, 2345, 12345\}$ and $C' = \{6, 7, \dots, n\}$. $\therefore |\mathcal{C}| = 16 \times 2^{n-5} = 2^{n-1}$.

In general, $\mathcal{D}_K = \{D \subseteq [n] : |D \cap [2k+1]| > k+1\}$. $\mathcal{D}_0 = \mathcal{A}^*$, $\mathcal{D}_1 = \mathcal{B}$, $\mathcal{D} = \mathcal{C}$: $\mathcal{D}$ is intersecting and $|\mathcal{D}| = 2^{n-1}$.

If $\mathcal{A} \subseteq \binom{[n]}{k}$ is intersecting, how large can it be? If $2k > n$ then $\binom{[n]}{k}$ is intersecting, so we hence

Note that one large intersecting family is $\mathcal{A}^* = \{A \in \binom{[n]}{k} : 1 \in A\}$, $|\mathcal{A}^*| = \binom{n-1}{k-1}$.



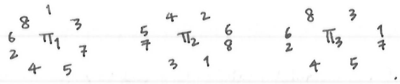
Theorem 3.7 (Erdős-Ko-Rado 1961)

If $2k \leq n$ and $\mathcal{A} \subseteq \binom{[n]}{k}$ is intersecting, then $|\mathcal{A}| \leq \binom{n-1}{k-1}$.

Proof – (Katona) let $n \geqslant 2k$ and $\mathcal{A} \subseteq \binom{[n]}{k}$ be intersecting. Let $\mathcal{C}_n$ be the family of cyclic permutations of $[n]$.

By this, we mean that two permutations of $[n]$ are considered the same if when written around a circle, we can form one from the other by rotation.

For instance, with $n=8$:

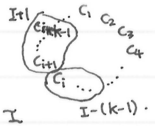


Here, $\pi_1 = \pi_2 \neq \pi_3$.

Then $|\mathcal{C}_n| = \frac{n!}{n} = (n-1)!$ We will continue this proof after introducing a lemma.

Given a cyclic permutation π and a set $A \in \mathcal{A}$, we say A is an interval in π if the elements of A appear consecutively.

$\boxed{\text{Lemma}}$ 3.8 If $\pi \in \mathcal{C}_n$ is a cyclic permutation of $[n]$ and $\mathcal{I} = \{I_1, \dots, I_t\}$ are intersecting intervals from π each of length k $(n \geq 2k)$, then $t \leq k$.

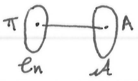


Proof – Let $I = \{c_i, c_{i+1}, \dots, c_{i+k-1}\} \in \mathcal{I}$. Note that I meets at most $2k-2$ other intervals, for π.

Namely: $I+1, I+2, \dots, I+(k-1)$ where $I+j = \{c_{i+j}, c_{i+j+1}, \dots, c_{i+j+k-1}\}$, $I-1, I-2, \dots, I-(k-1)$.

But $I+1$ and $I-(k-1)$ are disjoint, as are $I+j$ and $I-(k-1)$ for any $1 \leq j \leq k-1$. Hence $\exists$ at most one of $I+j$ and $I-(k-j)$ in $\mathcal{I}$ for each $1 \leq j \leq k-1$.

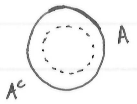
Thus, $|\mathcal{I}| \leq 1 + (k-1) = k$ // q.e.d.



(EKR, cont'd) Proof – Define a bipartite graph $G = (\mathcal{C}_n, \mathcal{A}; E)$. Join $\pi \in \mathcal{C}_n$ to $A \in \mathcal{A}$ iff A is an interval in π.

If $\pi \in \mathcal{C}_n$, then $d(\pi) = \#$ intervals of π that belong to $\mathcal{A}$. So $A \in \mathcal{A}$, then $d(A) = k!\,(n-k)!$

Double counting: $\sum_{\pi \in \mathcal{C}_n} d(\pi) = |E| = \sum_{A \in \mathcal{A}} d(A)$. $k|\mathcal{C}_n| \geq |E| = |\mathcal{A}|\, k!\,(n-k)!$



so $|\mathcal{A}| \leq \frac{k(n-1)!}{k!\,(n-k)!} = \binom{n-1}{k-1} = \frac{k}{n}\binom{n}{k}$ // q.e.d.

Note: $n > 2k \Rightarrow$ unique best family.

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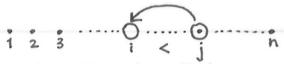
3.3 Compressions.

For $A \subseteq [n]$ and $1 \leq i < j \leq n$ we define the ij-compression of A to be $C_{ij}(A) = \begin{cases} A \setminus \{j\} \cup \{i\} & \text{if } j \in A,\ i \notin A \\ A & \text{otherwise.} \end{cases}$



e.g. if $A = 2\,4\,6$, $C_{34}(2\,4\,6) = 2\,3\,6$, $C_{34}(\overset{1}{\cancel{2}}\ \overset{2}{\cancel{3}}\ \overset{5}{\cancel{6}}) = 1\,2\,5$, $C_{34}(1\,2\,3) = 1\,2\,3$, $C_{34}(1\,3\,4) = 1\,3\,4$

Take out j, replace with i (compress to left)

We extend this to families of sets as follows: if $\mathcal{A} \subseteq \mathcal{P}([n])$, then the ij-compression of $\mathcal{A}$ is $C_{ij}(\mathcal{A}) = \{C_{ij}(A) : A \in \mathcal{A}\} \cup \{A \in \mathcal{A} : C_{ij}(A) \in \mathcal{A}\}$. (preserves size of family)

We say that a family $\mathcal{A} \subseteq \mathcal{P}([n])$ is compressed if $C_{ij}(\mathcal{A}) = \mathcal{A}$ for all $1 \leq i < j \leq n$. Generally speaking, compressions preserve whatever property the original family has.

e.g. Let $\mathcal{A} = \{\underline{1\,4}\,6, \underline{2\,3}\,6, \boxed{2\,4\,6}, \underline{1\,2\,4}\}$. $C_{34}(\mathcal{A}) = \{\overset{\text{comp}}{\underline{1\,3\,6}}^{*}, \underline{2\,3\,6}, \overset{\text{comp}}{\underline{1\,2\,3}}^{*}, \overset{\text{retained}}{\boxed{2\,4\,6}}\} = \mathcal{A}'$. $C_{26}(\mathcal{A}') = \{1\,2\,\overset{*}{3}, 2\,3\,6, 2\,4\,6, 1\,3\,6\}$. Following on,

$C_{16}(\mathcal{A}'') = \{1\,2\,3, 1\,2\,4, 1\,3\,6, 2\,3\,6\} = \mathcal{A}'''$. $C_{46}(\mathcal{A}''') = \{1\,2\,3, 1\,2\,4, 1\,3\,4, 2\,3\,4\} = \hat{\mathcal{A}}$. This is stable under compressions, we cannot compress any further.

Hence, we say that $C_{ij}(\hat{\mathcal{A}}) = \hat{\mathcal{A}}$, $\forall i,j$ where $1 \leq i < j \leq n$, and $\hat{\mathcal{A}}$ is (left)-compressed.

$\boxed{\text{Lemma}}$ 3.9 Let $\mathcal{A} \subseteq \binom{[n]}{k}$ and $1 \leq i \leq j \leq n$. Then

(i) $C_{ij}(\mathcal{A}) \subseteq \binom{[n]}{k}$, (ii) $|C_{ij}(\mathcal{A})| = |\mathcal{A}|$, (iii) If $\mathcal{A}$ is intersecting then so is $C_{ij}(\mathcal{A})$.

(iv) Repeatedly applying ij-compressions to family $\mathcal{A}$ will eventually yield a compressed family $\hat{\mathcal{A}}$, s.t. $\hat{\mathcal{A}} = C_{ij}(\hat{\mathcal{A}})$ $\forall 1 \leq i < j \leq n$.

Proof – (i),(ii) follow instantly from definition of C_{ij}. For (iii), suppose $\mathcal{A}$ is intersecting. Suppose $\exists A, B \in C_{ij}(\mathcal{A})$ s.t. $A \cap B = \phi$. $\Rightarrow$ not both A, B in $\mathcal{A}$. Since every "new" set in $C_{ij}(\mathcal{A})$ contains i, so A, B are not both "new".

WLOG, let $A \in \mathcal{A}$, $B \notin \mathcal{A}$. So $\exists\, C = (B \setminus \{i\}) \cup \{j\} \in \mathcal{A}$. Since $A \cap B = \phi$ and $A \cap C = \phi$, we must have $j \in A, i \notin A$. By definition, $D = C_{ij}(A) \in \mathcal{A}$, $D = (A \setminus \{j\}) \cup \{i\}$

so $C \cap D \subseteq (B \setminus \{i\}) \cap (A \setminus \{j\}) \subseteq A \cap B = \phi$, since $C, D \in \mathcal{A}$ // q.e.d.

For (iv), define $w(\mathcal{A}) = \sum_{A \in \mathcal{A}} \sum_{a \in A} a$. If $C_{ij}(\mathcal{A}) \neq \mathcal{A}$, then $w(C_{ij}(\mathcal{A})) \leq w(\mathcal{A}) - (j-i) \leq w(\mathcal{A}) - 1$ $\because$ $j > 1$. But $w(\mathcal{A}) \in \mathbb{N}$, so by well-ordering principle,

we eventually reach a compressed family // q.e.d.

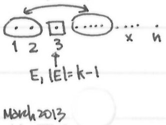
The theory of compressions allows us to provide a second proof of the Erdős-Ko-Rado theorem:

Proof – Use induction on $n \geq 2r$. $n = 2$ true, then for $n > 2$, let $\mathcal{A} \subseteq \binom{[n]}{k}$ be intersecting. If $n = 2k$, then $\binom{n-1}{k-1} = \frac{1}{2}\binom{n}{k}$ and $(A \in \mathcal{A} \Rightarrow A^c \notin \mathcal{A}) \Rightarrow |\mathcal{A}| \leq \frac{1}{2}\binom{n}{k}$ true.

Then suppose $n \geq 2k+1$. Now by applying compressions, suppose $\mathcal{A}$ is compressed. Let $\mathcal{B} = \{A \in \mathcal{A} : n \notin A\}$, $\mathcal{C} = \{A \in \mathcal{A} : n \in A\}$. This is a partition of $\mathcal{A}$, i.e.

$\mathcal{A} = \mathcal{B} \dot{\cup} \mathcal{C}$. $\mathcal{B} \subseteq \binom{[n-1]}{k}$. By inductive hypothesis, $n \geq 2k+1 \Rightarrow |\mathcal{B}| \leq \binom{n-1-1}{k-1} = \binom{n-2}{k-1}$. NTP: $\binom{n-2}{k-2}$ is an upper bound for $|\mathcal{C}|$.

Consider $\mathcal{D} = \{C \setminus \{n\} : C \in \mathcal{C}\}$, so $\mathcal{D} \subseteq \binom{[n-1]}{k}$. If we show that $\mathcal{D}$ is intersecting, then our inductive hypothesis $\Rightarrow$ $\mathcal{D} \leq \binom{n-1-1}{k-1-1} = \binom{n-2}{k-2}$.



Suppose $D, E \in \mathcal{D}$ s.t. $D \cap E = \phi$. Then $D \cup \{n\}, E \cup \{n\} \in \mathcal{A}$. Since $|D| = k-1 = |E|$, D, E disjoint. Then since $n \geq 2k+1$, $\exists\, x \in [n-1] \setminus (D \cup E)$

Since $\mathcal{A}$ is compressed, $C_{xn}(D) = (D \setminus \{n\}) \cup \{x\} \in \mathcal{A}$. But $C_{xn}(D) \cap (E \cup \{n\}) = \phi \Rightarrow$ contradiction, since $\mathcal{A}$ is intersecting // q.e.d.

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3.4 Linear Algebra Method.

This provides a different perspective to problems in Graph Theory.

Lemma 3.10 If $\underline{v}_1, \underline{v}_2, \cdots, \underline{v}_m \in V$, V is a vector space of dimension d and $\underline{v}_1, \dots, \underline{v}_m$ are linearly independent, then $m \leq d$.

Linearly independent: let $\underline{v}_1, \dots, \underline{v}_m \in V$, V is a vector space over field $\mathbb{F}$ are LI if $\sum_{i=1}^{m} \lambda_i \underline{v}_i = 0 \Rightarrow \lambda_i = 0 \; \forall i$.

Theorem 3.11 If $\mathcal{A} = \{A_1, \dots, A_m\} \subseteq \mathcal{P}([n])$ with $|A_i|$ is odd $\forall i$, and $|A_i \cap A_j|$ is even $\forall i \neq j$, then $m \leq n$.

Proof - For $A_i \in \mathcal{A}$, consider its incidence vector $\underline{v}_i \in \mathbb{F}_2$, the field with 2 elements. $(\underline{v}_i)_j = \begin{cases} 1, & j \in A_i \\ 0 & \text{otherwise} \end{cases}$ e.g. $n=6$, $A = \{1\,3\,5\}$, $\underline{v}_2 = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}$.

So we have m vectors $\underline{v}_1, \dots, \underline{v}_m$. Consider $\underline{v}_i \cdot \underline{v}_j = \sum_{k=1}^{n} v_{ik} v_{jk} = |A_i \cap A_j| = \begin{cases} 1 & i=j \\ 0 & i \neq j \end{cases}$. Hence $\underline{v}_1, \dots, \underline{v}_m$ are orthogonal $\Rightarrow$ they are LI.

Hence, from lemma 3.10, $m \leq \dim(\mathbb{F}_2{}^n) = n$ // q.e.d.

Theorem 3.12 (Fisher's Inequality, 1940).

If $\mathcal{A} = \{A_1, \dots, A_m\} \subseteq \mathcal{P}([n])$ and $\exists\, 1 \leq k \leq n$ s.t. $\forall i \neq j$, $|A_i \cap A_j| = k$, then $m \leq n$.

Proof - Let $\mathcal{A}$ be given with above properties. For $A_i \in \mathcal{A}$, let $\underline{v}_i$ be its incidence vector, $\underline{v}_i \in \mathbb{R}^n$. $(\underline{v}_i)_j = \begin{cases} 1 & j \in A_i \\ 0 & \text{otherwise} \end{cases}$. We want to show that $\{\underline{v}_1, \dots, \underline{v}_m\}$ is LI. Suppose for a contradiction, $\exists\, \lambda_1, \dots, \lambda_m \in \mathbb{R}$ not all zero with $\sum_{i=1}^{m} \lambda_i \underline{v}_i = \underline{0}$. Note that $\underline{0} \cdot \underline{0} = 0$. Hence,

$$0 = \underline{0} \cdot \underline{0} = \left(\sum_{i=1}^{m} \lambda_i \underline{v}_i\right) \cdot \left(\sum_{j=1}^{m} \lambda_j \underline{v}_j\right) = \sum_{i=1}^{m} \lambda_i^2 \underline{v}_i \cdot \underline{v}_i + \sum_{i \neq j} \lambda_i \lambda_j \underline{v}_i \cdot \underline{v}_j = \sum_{i=1}^{m} \lambda_i^2 |A_i| + k \sum_{i \neq j} \lambda_i \lambda_j = \underbrace{\sum_{i=1}^{m} \lambda_i^2 (|A_i| - k)}_{①} + \underbrace{k \left(\sum_{i=1}^{m} \lambda_i\right)^2}_{②}.$$

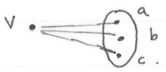
$k > 0$, $\left(\sum_{i=1}^{m} \lambda_i\right)^2 \geq 0 \Rightarrow \sum_{i=1}^{m} \lambda_i^2 (|A_i| - k) \leq 0$. But $\lambda_i^2 \geq 0$ and $|A_i| - k \geq 0$, so ① = ② = 0. ① = 0 $\Rightarrow |A_i| \neq k \Rightarrow \lambda_i = 0$. Also, since $|A_i \cap A_j| = k$, then $|A_i| \geq k$ $\forall i$ with equality at most once. Hence, all but 1 λ_i must be 0. ② = 0 $\Rightarrow \sum_{i=1}^{m} \lambda_i = 0 \Rightarrow$ impossible, since exactly one λ_i is non-zero. $\Rightarrow$ contradiction $\Rightarrow \{\underline{v}_1, \dots, \underline{v}_m\}$ is LI. $m \leq \dim(\mathbb{R}^n) = n$ // q.e.d.

chapter 4
RAMSEY THEORY.

Ramsey theory is a study of order.

Let $s, t \geq 2$ be integers. Then we define the <u>Ramsey number</u> $R(s,t) = \min\{n :$ whenever K_n has its edges coloured red and blue, always $\exists$ a red K_s or blue $K_t\}$.

This is not yet a definition, as we have yet to show that the set is non-empty.



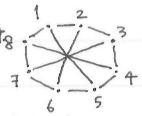
Proposition 4.1 $R(3,3) = 6$.

Proof - (i) NTP: $R(3,3) \leq 6$. Take a red-blue colouring of the edges of K_6. Take a vertex $v \in V(K_6)$. Since $d(v) = 5$, wlog v is incident to at least 3 red edges (or by symmetry, blue), with endpoints a, b, c. Either at least one of ab, ac, bc is red $\Rightarrow$ we get a red K_3, or they are all blue $\Rightarrow$ we have a blue $K_3 \Rightarrow R(3,3) \leq 6$. $\because$ colouring K_6 always works.

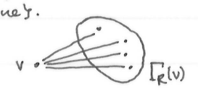
(ii) NTP: $R(3,3) > 5$. We need to find a colouring of K_5 s.t. neither red K_3 nor blue K_3 exists. With the colouring on the right, $R(3,3) > 5$. Hence $R(3,3) = 6$ // q.e.d.

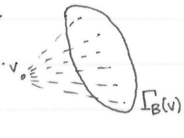
Proposition 4.2 $R(3,4) = 9$.

Proof - (i) NTP: $R(3,4) > 8$. Consider the colouring as on right: taking $V(K_8) = [8]$. red edges $= \{i\; i{+}1 : 1 \leq i \leq 8\} \cup \{i\; i{+}4 : 1 \leq i \leq 4\}_8$. All other edges are blue. No red K_3 and no blue K_4.



(ii) NTP: $R(3,4) \leq 9$. Take a red-blue colouring of K_9. Let $v \in V(K_9)$. $\Gamma_R(v) = \{w : vw \text{ is red}\}$, $\Gamma_B(v) = \{w : vw \text{ is blue}\}$. Also define $d_R(v) = |\Gamma_R(v)|$, $d_B(v) = |\Gamma_B(v)|$. So $d_R(v) + d_B(v) = d(v) = 8$. If $\exists\, v \in V(K_9)$ with $d_R(v) \geq 4$, then either $\Gamma_R(v)$ contains a red edge (red K_3), or it consists entirely of blue edges (blue K_4). WLOG, we assume $d_R(v) \leq 3$ $\forall v \in V(K_9) \Rightarrow d_B(v) \geq 5$. If $\exists\, v \in V(K_9)$ s.t. $d_B(v) \geq 6 = R(3,3) \Rightarrow \Gamma_B(v)$ contains red K_3 or blue K_3. $\Rightarrow K_9$ contains red K_3 or a blue K_4. Finally, consider $d_B(v) = 5$ $\forall v \in V(K_9)$; the only remaining case.





But by Handshake Lemma, $\sum_{v \in V(K_9)} d_B(v) = 2 \times \#$blue edges. But $\sum_{v \in V(K_9)} d_B(v) = 9 \cdot 5 = 45 \Rightarrow$ impossible, since $2 \nmid 45$. Hence this case does not exist.

Overall then, $R(3,4) = 9$. // q.e.d.

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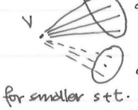
We also have to establish that the set is non-empty, such that the Ramsey number is well-defined.



either
$d_R(v) \geq R(s-1,t)$
or
$d_B(v) \geq R(s,t-1)$.

Theorem 4.3 (Ramsey 1930).

If $s,t \geq 2$ then $R(s,t)$ is finite and satisfies $R(s,t) \leq \binom{s+t-2}{s-1}$.

Proof - By induction on $s+t$. Note that $R(2,t) = t$, $R(s,2) = s \Rightarrow$ result holds if $s+t$ is 2. So now suppose $s,t \geq 3$ and the result holds for smaller $s+t$.

Let $n = R(s-1,t) + R(s,t-1)$. This exists, by our inductive hypothesis. We claim that $R(s,t) \leq n$. Take a red-blue colouring of the edges of K_n. Let $v \in V(K_n)$. Define $\Gamma_R(v) = \{w : vw \text{ is red}\}$, $d_R(v) = |\Gamma_R(v)|$, $\Gamma_B(v) = \{w : vw \text{ is blue}\}$, $d_B(v) = |\Gamma_B(v)|$. So $d_R(v) + d_B(v) = d(v) = n-1$. Since $n = R(s-1,t) + R(s,t-1)$, we must have $d_R(v) \geq R(s-1,t)$ or $d_B(v) \geq R(s,t-1)$ [otherwise, $d(v) \leq n-2$]. WLOG, suppose first case holds: then either $\Gamma_R(v)$ contains a red K_{s-1}, which together with v forms a red K_s, or $\Gamma_R(v)$ contains a blue K_t. Similar argument applies for second case.

Hence, $R(s,t) \le n = R(s-1,t) + R(s,t-1) \le \binom{s-1+t-2}{s-1-1} + \binom{s+t-1-2}{s-1} = \binom{s+t-2}{s-1}$ // q.e.d.

Proposition 4.4 $R(4,4) = 18$.

Proof – Claim: $R(4,4) > 17$. Recall that we say x is a quadratic residue mod n if $\exists\, y$ s.t. $x \equiv y^2$ mod n. Let $n = 17$, and colour edges of K_{17} as follows:

$V(K_{17}) = \mathbb{Z}/17 = \{0, 1, \ldots, 16\}$. We colour xy red $\Leftrightarrow$ $x-y$ is a quadratic residue mod 17 (the graph formed by red edges is the Paley graph).

All other edges are blue. We can check theoretically that there is red K_4 and no blue K_4.

We know that $R(4,4) \le R(3,4) + R(4,3) = 9+9 = 18$ (using proof of Theorem 4.3, and $R(3,4) = 9$) // q.e.d.

This is the last known value of $R(k,k)$. $R(5,5)$ is not known, although we have bounds $43 \le R(5,5) \le 49$.

Theorem 4.5 (Conlon 2009).

There exists a constant $c > 0$ s.t. $R(s,s) \le \frac{1}{s^{c \log s / \log \log s}} \binom{2s-2}{s-1}$.

Proof – omitted.

Note: this is the first improvement in the bounds for Ramsey numbers in something like 70 years.

We will jump ahead to show something more general:

Theorem 4.12 Let $s_1, s_2, \ldots, s_k \ge 2$ define $R_k(s_1, s_2, \ldots, s_k) = \min\{n :$ whenever the edges of K_n are coloured with colours $c_1, c_2, \ldots, c_k$, $\exists$ a c_i-coloured K_{s_i} for some $1 \le i \le k\}$

Then $\forall\, k \ge 2$ and $s_1, s_2, \ldots, s_k \ge 2$, $R_k(s_1, s_2, \ldots, s_k)$ is finite.

Proof – Induction on number of colours, k. By Ramsey's Theorem, this is true for $k=2$. So let $k \ge 3$. Let $n = R_{k-1}(s_1, s_2, \ldots, s_{k-2}, \overbrace{R(s_{k-1}, s_k)}^{\text{"colourblind in last 2"}})$. ($\exists$ by ind. hypothesis).

We claim that $R_k(s_1, \ldots, s_k) \le n$: take a colouring of edges on K_n with colours $c_1, \ldots, c_k$. Now suppose we cannot distinguish between colours c_{k-1} and c_k. Then we have a colouring of edges of K_n with $k-1$ colours: $c_1, c_2, \ldots, c_{k-2}$ and "c_{k-1} or c_k". By definition of $R_{k-1}(s_1, \ldots, s_{k-2}, R(s_{k-1}, s_k))$,

we either have a c_i-coloured K_{s_i} for some $1 \le i \le k-2$, or we have a copy of $K_{R(s_{k-1}, s_k)}$ coloured with colours c_{k-1} and c_k.

But then, Ramsey's theorem implies that this contains a c_{k-1}-coloured $K_{s_{k-1}}$, or a c_k-coloured K_{s_k}.

We note $R_k(s) = R_k(\overbrace{s, s, \ldots, s}^{k})$

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We also want to find a lower bound for $R(s,s)$: i.e.

Theorem 4.6 (Erdős 1947)

If $n \ge s \ge 2$ satisfy $\binom{n}{s} 2^{1-\binom{s}{2}} < 1$, — ⊛ then $R(s,s) > n$.

Proof – Let n, s satisfy ⊛. We need to prove $\exists$ a red-blue colouring of the edges of K_n with no monochromatic K_s.

Define a random colouring as follows: Flip independent fair coins for each edge. If coin is Heads, colour edge red; Tails, colour edge blue.

Consider $X = \#$ of monochromatic copies of K_s. If $\mathbb{E}[X] < 1$, then $\exists$ a colouring with no monochromatic K_s. Hence, $R(s,s) > n$. Prove this:

Fix $A \subset V(K_n)$, $|A| = s$. Let $X_A = \begin{cases} 1 & A \text{ forms monochromatic } K_s \\ 0 & \text{otherwise.} \end{cases}$ $P(X_A = 1) = P(\text{all edges between vertices in } A \text{ are red}) + P(\text{all edges between vertices in } A \text{ are blue})$

Hence, $P(X_A = 1) = 2 \cdot \frac{1}{2^{\binom{s}{2}}} = 2^{1-\binom{s}{2}}$. Then $X = \sum_{\substack{A \subset V(K_n) \\ |A| = s}} X_A \Rightarrow \mathbb{E}[X] = \sum_{\substack{A \subset V(K_n) \\ |A| = s}} P(X_A = 1)$ by linearity of expectation. $= \binom{n}{s} \frac{2}{2^{\binom{s}{2}}} < 1$. by ⊛.

Corollary 4.7 If $s \ge 2$, then $R(s,s) \ge 2^{s/2}$.

Proof – $R(2,2) = 2$, $R(3,3) = 6 \ge 2^{3/2}$. Let $s \ge 4$ and $n = \lfloor 2^{s/2} \rfloor$. We need to show that ⊛ holds: $s! > 2^s \Rightarrow \binom{n}{s} \frac{2}{2^{\binom{s}{2}}} < \frac{n^s}{2^s} \frac{2}{2^{\binom{s}{2}}} \le \frac{2^{\frac{s^2}{2}+1}}{2^{\frac{s^2}{2}+\frac{s}{2}}} = \frac{1}{2^{s/2-1}} \le \frac{1}{2} < 1$ // q.e

Overall, this gives us $2^{s/2} \le R(s,s) \le \frac{4^s}{s}$.

Theorem 4.8 If $n \ge 3$, there are no nontrivial integer solutions to $x^n + y^n = z^n$.

Proof – Obviously omitted. ("left as exercise" by Dr. Talbot).

Theorem 4.9 For every $n \ge 1$, there exists p_n s.t. if $p \ge p_n$ is prime, the congruence $x^n + y^n \equiv z^n \pmod p$ has non-trivial solutions.

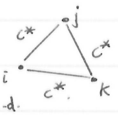
Theorem 4.10 (Schur 1916).

For any $k \ge 1$, $\exists\, S(k)$ s.t. in any k-colouring of the integers $\{1, 2, \ldots, S(k)\}$, there is a monochromatic solution to $u + v = w$ (i.e. u, v, w all same colour).

Proof – Recall $R_k(3) = \min\{n :$ Every k-colouring of the edges of K_n contains a monochromatic $K_3\}$. Set $n = R_k(3)$. Consider a k-colouring of $\{1, 2, \ldots, n\}$

called c. Define a k-colouring of the edges of K_n (with $V(K_n) = \{1, 2, \ldots, n\}$). For $ij \in E(K_n)$, $i < j$, $c'(ij) = c(j-i)$.

By definition of $R_k(3)$, there is a monochromatic K_3. Say with vertices $i < j < k$, $c'(ij) = c'(ik) = c'(jk) = c^*$.



$\Rightarrow c(\overset{u}{j-i}) = c(\overset{w}{k-i}) = c(\overset{v}{k-j}) = c^*$. $\Rightarrow$ $u + v = w$ and $c(u) = c(v) = c(w) = c^* \Rightarrow S(k)$ is well-defined, and satisfies $S(k) \le n = R_k(3)$ // q.e.d.

Lemma 4.11 If p is prime, then $\mathbb{Z}_p^* = \{1, 2, \dots, p-1\}$ is a cyclic group. i.e. $\exists g \in \mathbb{Z}_p^*$ s.t. $\{g^1, g^2, \dots, g^{p-1}\} = \mathbb{Z}_p^*$.

Proof – omitted.

Aside: $u+v=w$, $u = g^{m_u}$, $v = g^{m_v}$, $w = g^{m_w}$.
For any m $\exists c$ s.t. $m_u = a_u n + c_u$, $0 \leq c_u \leq n$
$u = g^{a_u n + c_u}$, $v = g^{a_v n + c_v}$, $w = g^{m_w + c_w}$
If $c_i = 0 \Rightarrow$ solution. If c_i non-zero but all the same, require monochromatic solutions.

(Theorem 4.9) Proof – let $n \geq 1$ be given. Take $p \geq S(n)$, given by Schur's theorem, with p prime. By Lemma 4.11, $\exists$ generator g for $\mathbb{Z}_p^*$.
For each $x \in \mathbb{Z}_p^*$, $\exists m$ s.t. $x = g^m \bmod p$. Now define colour for x by $c(x) = i$, where

$m = an + i$, $0 \leq 1 \leq n-1$. (i.e. colour is the remainder upon division by n). So we have an n-colouring of $\{1, 2, \dots, p-1\}$.

Since $p-1 \geq S(n)$, $\exists u, v, w$ s.t. $u+v=w \Rightarrow c(u) = c(v) = c(w) = c$. $\therefore u = g^{a_u n + c}$, $v = g^{a_v n + c}$, $w = g^{a_w n + c}$.

Let x be $x = g^{a_u}$, $y = g^{a_v}$, $z = g^{a_w}$, then ${}_{\wedge}^{\text{in mod } p,}$ $x^n + y^n \equiv u g^{-c} + v g^{-c} \equiv g^{-c}(u+v) = g^{-c} w = g^{a_w n} = z^n$, q.e.d.

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Theorem 4.13*. (Green-Tao 2007).

The primes contain arbitrary long arithmetic progressions.

Note: The longest known chain now is of length 26.

Proof – omitted, this follows from vdW theorem:

Theorem 4.13 (van der Waerden 1927).

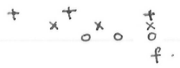
1 3 5 6 78 } three APs, all with next term 9.

$\forall t, k \geq 1$, $\exists W(t,k) \in \mathbb{Z}$ s.t. every k colouring of $[W(t,k)]$ contains a monochromatic (MAP) AP of length t.

Definition If $P_1, \dots, P_r$ are MAPs s.t. each are of a different colour, and with the property that the next term in each P_i is the same, say f; then we say $P_1, \dots, P_r$ are <u>colour-focused</u> APs (CFAPs) with <u>focus</u> f.

Proof – By nested induction, first on t. $W(1,k) = 1$. $W(2,k) = k+1$, since if we colour $[k+1]$ with k colours, some colour is used twice $\Rightarrow$ MAP length 2.
So now let $t \geq 3$, and suppose $W(t-1, k)$ exists for all choices of k. claim: For r s.t. $1 \leq r \leq k$, $\exists n_r(t,k)$ s.t. if $[n_r(t,k)]$ are k-coloured, $\exists$ either (a) a monochromatic AP of length t or (b) $\exists r$ CFAPs of length t-1. If this claim holds – i.e. claim is true, with $r = k$. If we k-colour $[n_k(t,k)]$, then either we have (a) ✓ or (b) $\exists k$ CFAPs of t-1: i.e. $P_1, \dots, P_k$ are CFAPs of length t-1. $\Rightarrow$ one of the P_is has the same colour as the common focus $\Rightarrow \exists$ a monochromatic AP of length t. Hence, we can take $W(t,k) = n_k(k,t)$. So now we just need to prove the claim:

Proof of claim: By induction on r. For $r=1$, take $n_1(t,k) = W(t-1,k)$. Now suppose $2 \leq r \leq k$ and ${}_{\wedge}^{n=}$ $n_{r-1}(t,k)$ exists. Let $n_r(t,k) = W(t-1, k^{2n})2n$.
Take a k-colouring of $[W(k-1, t^{2n})2n]$, where $n = n_{r-1}(t,k)$. Assume $\nexists$ MAP of length t. Then $[W(t-1, k^{2n}) \cdot 2n] = B_1 \dot\cup B_2 \dot\cup \dots \dot\cup B_{W(t-1,k^{2n})}$, a collection of blocks defined as $B_1 = \{1, \dots, 2n\}$, $B_2 = \{2n+1, \dots, 4n\}$ etc. Each B_i has been coloured with k colours: $\therefore$ there are k^{2n} different ways that a block could be coloured. By definition of $W(k-1, t^{2n})$, we have $B_s, B_{s+v}, B_{s+2v}, \dots, B_{s+(t-2)v}$ are identically coloured blocks. since $W(t-1, k^{2n})$ means the <u>subscripts</u> contain a MAP. Each block B_i has length $2n_{r-1}(t,k)$. $\therefore$ each B_i contains $P_1, \dots, P_{r-1}$ colour focused APs of length t-1. (true even for length $n_{r-1}(t,k)$). But B_i has length $2n_{r-1}(t,k)$, so their focus is also contained within!



Then we have $P_i = a_i, a_i + d_i, a_i + 2d_i, \dots, a_i + (t-2)d_i$, $1 \leq i \leq r-1$. Common focus is f.

Since Bs and Bs+v, Bs+2v, …, Bs+(t-2)v are all identically coloured, we define $P_i' = a_i, a_i + (d_i + 2nv), a_i + 2(d_i + 2nv), \dots, a_i + (t-2)(d_i + 2nv)$. Clearly there are all MAPs of length t-1 and different colours. Hence, focus is $f + (t-1) \cdot 2nv$. This gives us r-1 CFAPs. For the final one, moreover, set $P_r' = f, f+2nv, f+4nv, \dots, f+(t-2)nv$, this is another MAP of length t-1, and a different colour to $P_1', \dots, P_{r-1}'$. So $P_1', \dots, P_r'$ are r CFAPs of length t-1 with common focus $f + (t-1)2nv$.
$\therefore$ setting $n_r(r,k) = W(t-1, k^{2n}) \cdot 2n$ will do, q.e.d.

END OF SYLLABUS.